University of Alberta

Control of Systems Described by Multiple Models

by

Dejan Kihas

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science

in

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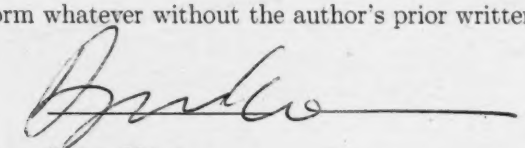

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Date: April 15, 2003

"Machines that walk, swim, or fly are gain scheduled".[1]

[1]W.J. Rugh, J.S. Shamma from "Research on Gain Scheduling", Automatica 36(2000)

Abstract

The problem of controlling a nonlinear system using multiple linear approximations over a large operating range, referred to as the "multiple model approach to control systems", is analyzed in this thesis. First, *the stability* of this class of systems is analyzed in detail. The concept of stable motion and the theory of input-to-state stability are explained as the basis of the required stability analysis. Second, it is explained that for a nonlinear system it is necessary to define regions of operation in which stability and performance are guaranteed. Trough the determination of these regions, it is possible to describe the system by a set of local multiple models preserving those guarantees. This can be accomplished by modeling the original nonlinear system as a linear time invariant system plus an uncertainty term. This uncertainty term should reflect the maximal deviation between the true trajectories of the original nonlinear model and those predicted by the linear approximation. Hence, it can be considered as a "measure of nonlinearity". In this way, a measure of nonlinearity describes the mismatch between the state signals of a nonlinear system and its linear model at an operating point. In this

thesis, two useful measures of nonlinearity are defined: one describes the uncertainty in the $\mathcal{L}_\infty$ sense, and the other in the $\mathcal{L}_2$ sense. Techniques for estimating regions where these measures satisfy some desired requirements are also given. Both norms are very useful in the development of the multiple model. The control based on multiple models offers an alternative to gain scheduling and a solution to its drawback regarding stability issues.

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I thank my friends from the Systems and Control Research Lab for sharing my happiness and understanding my sorrow. I hope we will exchange many postcards in the future.

To my Loving Grandparents

Juraj and Magdalena

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Chapter 1

Introduction

Researchers have been interested in control of nonlinear systems for a very long time. Progress in nonlinear control design, however, has been difficult because of the intrinsic complexity of the problem. In general, nonlinear control methods are complex and can be applied only to a narrow class of systems. For example, methods such as backstepping and feedback linearization [29], [36], [58], [1], [48], [49], [67], [68] can be applied to nonlinear systems with some specific structure, but not to arbitrary nonlinear systems. Thus, nonlinear control methods cannot serve all needs of real industrial control problems. Very often in industry such as the chemical, automotive and aeronautical, nonlinear models are too complex to be used for controller design. Linear models and associated techniques for linear control design are typically used to control the plant under certain

specific operating conditions [32], [69], [12], [63], [40], [39], [41], [42]. This type of control is only valid in a small region around the operating point.

One way to approach the control of a nonlinear system in a wide range of conditions is to linearize the model at a number of operating points, and then design one linear feedback controller at each operating point. These *local* controllers can then be "switched" or "scheduled" as the system changes operating conditions. One popular engineering method, based on this idea, is the so-called gain scheduling technique [43], [75], [11], [37], [38], [33], [29], [51], [52], [53], [27], [28], [76], [21]. In this method the parameters of the controller are scheduled according to changes in system dynamics. Unfortunately, this technique evolved with little theoretical support. Indeed, there are no general rules for designing gain scheduling regulators and a general approach has not been developed, except for a few guidelines or rules of thumb such as "the scheduling variable should vary slowly" and "the scheduling variable should capture plant's nonlinearities" [37]. An elementary approach to gain scheduling, along with several examples are given in [43]. In the work of Rugh [75], [11], gain scheduling is considered based on a family of linear dynamic controllers designed for a family of linear models derived by linearization of the plant about constant operating points. A necessary and sufficient condition for the existence of gain scheduled controllers satisfying certain requirements is presented. In [37], [38], conditions are given that confirm and formalize the popular

rules of thumb regarding gain scheduled design. It is shown that basic guidelines for choosing the scheduling variable indeed have a rigorous mathematical justification.

One of the main problems when working with the control of a nonlinear plant based on multiple linear approximations is how to ensure stability of the overall design. Indeed, local models carry only "local" information about the system's dynamics, and some form of mechanism that ensures global, or semi-global stability must be found. A novel approach to this problem, referred to as an input - output approach to systems described using multiple models was introduced in [27], [28]. These articles introduce the concept of *stable motion*, where the idea is to define a set of multiple models, locally input - output stable in the vicinity of each operating point, whose regions of stability overlap neighboring models. Therefore stability remains a local concept, but safe transitions can be performed. The main contribution of this approach is the introduction of the concept of the stable motion to investigate control of systems in a wide range of operating points. Also, in the same work it is suggested to base the design of a gain scheduling control on the given stability framework. Some related results have also been reported in [51], [52], [53]. These references, focus of stability in the sense of Lyapunov and present an algorithm to obtain generalized Lyapunov functions for global stability. Their solution is conservative in that it relies on the successful construction of a suitable Lyapunov function.

It is evident that the attempt to solve the problem of controlling a complex nonlinear system in a wide range of conditions by linearizing the model at a number of operating points and then designing a linear feedback controller for each operating point leads us to trouble due to stability reasons. A linear controller can stabilize the system in a small region about the operating point. Hence, the issues of stability and performance become a local concept and we have to establish a system of parameters and measures that will describe the regions where the system is stable and where certain performance are guaranteed. These parameters and measures are referred to as *Measures of Nonlinearity* and represents the quantified difference between the behavior of a nonlinear system and its linearized model at an operating point.

Motivated by these problems that arise when designing gain scheduling controllers, in this thesis we focus on nonlinear models and their linearization about an equilibrium point. Considering the fact that the accuracy of a linear approximation deteriorates as the region of operation about the equilibrium point becomes large, we look for a measure of nonlinearity that takes explicit account of the region of operation in the state space around the equilibrium point. We propose two new measures of nonlinearity:

(i) we first view both the nonlinear system and its linearization as mappings from input-to-state. We define the error between the state of the original nonlinear

system and that of the linearization and find the region of the state space where this error is norm-bounded. We cast our solution in the context of the input-to-state theory of systems initiated by Sontag [15], and compute the maximum input size that guarantees that the error remains bounded by an arbitrary pre-selected value.

(ii) In our second solution we study the same problem but define our measure of nonlinearity of the state error in the integral-square (or $\mathcal{L}_2$-norm) sense.

Both measures are found to be very useful in robust control design, and they provide the necessary elements for the design of gain scheduling controllers. Using these measures we are able to describe regions in the state space where the error, in the sense defined above, is less than a specified value. Furthermore, we are able to define bounds on input signal such that state variables do not leave the specified regions. By choosing these regions to be connected we cover an extended region over the state space where the system under control is both stable and its error signal, *i.e.* its performance, is guaranteed under certain conditions.

The rest of this thesis is organized as follows: In Chapter 2, we introduce our notation as well as some necessary background on model representation. Chapter 2 is based on [29], [64], [9], [27], [28]. In Chapter 3, we summarize the basics of Lyapunov stability theory, input-output stability, and input-to-state stability to be used through-

out the rest of this thesis [29], [36], [74], [65], [15], [16], [17], [64], [22], [23], [24], [54]. We also review the notion of stable motion that will play an important role in this work, [27], [28]. Chapter 4 gives a survey on control techniques and explains the importance of using gain scheduling, its virtuous and drawbacks. Chapter 4 is based on [21], [43], [11], [75], [37], [38], [51], [52], [53], [33], [29], [27], [28], [45], [50], [47], [67], [68], [48], [49], [10], [72], [76], [58], [32], [69], [12], [63], [73], [39], [41], [42]. Nonlinearity measures are studied in Chapter 5. After some discussion and literature review (based on [20], [55], [56], [57], [59], [60], [66], [3], [4], [5], [6], [7]), two new measures of nonlinearity are introduced:

(i) First, we view both the nonlinear system and its linearization as mappings from input-to-state. We define the error between the state of the original nonlinear system and that of the linearization and find the region of the state space where this error is bounded in the sense of $\mathcal{L}_\infty$-norm. We cast our solution in the context of the input-to-state theory of systems, and compute the maximum input size that guarantees that the error remains bounded by an arbitrary pre-selected value.

(ii) In our second solution we study the same problem but define our measure of nonlinearity of the state error in the integral-square (*i.e.* $\mathcal{L}_2$-norm) sense.

Chapter 6 contains a case study. Namely, we consider the nonlinear model of a DC motor [80], and compute the measures of nonlinearity defined in Chapter 5. The

importance of these measures in the context of control systems using multiple models is discussed in Chapter 7. Finally, Chapter 8 some conclusions and directions for future work.

Chapter 2

Background and Notation

In this section we introduce our notation as well as the main tools to be used throughout the rest of this thesis. See references [29], [64], [9], [27], [28] for further details.

2.1 Norms in a Linear Space

The space $\mathcal{L}_p^m$ for $1 \leq p \leq \infty$ is defined as the set of all piecewise continuous functions $u : [0, \infty) \rightarrow R^m$ such that

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p dt \right)^{1/p} < \infty$$

where $\| \cdot \|$ denotes the p euclidian norm.

The extension of the space $\mathcal{U}$, denoted as $\mathcal{U}_e$, is defined as the space of all signal functions $u(t)$ whose truncation u_T belongs to $\mathcal{U}$. The truncation is defined as follows:

$$u_T(t) = \begin{cases} u(t), & if\ t \leq T \\ 0, & if\ t > T \end{cases}$$

2.2 System Models

Mathematical models of physical systems usually take the form of state-space realizations or input-output descriptions. State-space realizations originated from the classical theory of dynamics of particles and rigid bodies (e.g. [45], [25]), and play a fundamental role in modern control theory. A state space realization of a physical system is the most frequently represented as a set of ordinary coupled differential equations of the first order which represents their behavior in time:

$$\dot{x}_1 = f_1(x_1, ..., x_n, u_1, ..., u_p)$$

$$\dot{x}_2 = f_2(x_1, ..., x_n, u_1, ..., u_p)$$

...

$$\dot{x}_n = f_n(x_1, ..., x_n, u_1, ..., u_p)$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t, and $u_1, u_2, ..., u_p$ are specified input variables. The variables $x_1, x_2, ..., x_n$ are called the state variables. To write the above equations in a compact form we use vector notion. Thus, we define:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_p \end{bmatrix}, \quad f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x,u) \end{bmatrix}$$

and obtain the following compact form:

$$\dot{x} = f(x,u)$$

where $x \in \mathcal{D} \subseteq R^n$, $u \in \mathcal{U} \subseteq R^p$. This equation is called the state equation, x is the *state* vector, and u is the *input* vector of the system. The output equation takes the following form:

$$y = h(x,u)$$

which defines a q-dimensional *output* vector.

An input-output model focuses on the relationship between inputs and outputs and can be represented as a mapping or operator $\mathcal{H}$ from one space of signals (functions in time) to another:

$$y_T = \mathcal{H}u_T$$

where y_T is the output signal from a normed linear space $\mathcal{L}_e^q$ of function $y : [0, \infty) \rightarrow R^q$; and u_T is the input signal from a normed linear space $\mathcal{L}_e^m$ of function $u : [0, \infty) \rightarrow R^m$, and $\mathcal{H}$ is a operator (not necessarily linear) that maps u_T into y_T.

Input-output models can be obtained from state space realizations in the obvious manner, *i.e.*, if

$$\begin{array}{c} \dot{x}(t) = f(x(t), u(t)) \\ y = h(x(t), u(t)) \end{array}, \tag{2.1}$$

with $f : \mathcal{D} \times \mathcal{U} \rightarrow R^n$ and $h : \mathcal{D} \times \mathcal{U} \rightarrow R^p$. Then, the input output system $\mathcal{H}$ is given by

$$\Sigma(\mathcal{H}) \left\{ \begin{array}{l} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{array} \right. . \tag{2.2}$$

When talking about x as state space variable, we say that $x \in \mathcal{D}$. Also, when talking about u as input system variable, we say that $u \in \mathcal{U}$. The state space variable $x(t)$ and

input system variable $u(t)$ are changing in time and also are treated as signals, members of some signal normed space. The reason for this is the need to analyze variables x, u and y both as signals of some signal normed space and elements of some set in the Euclidian vector space.

In this thesis we will focus on nonlinear systems of the form:

$$\Sigma(\mathcal{F}) \begin{cases} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{cases} . \tag{2.3}$$

Nonlinear systems of this form are called input-affine systems and include many cases of the practical importance.

2.3 Multiple Model Representation

We now introduce the terminology and notation related to systems represented by multiple models.

Definition 2.3.1 *Operating point*: A pair of real numbers $(x_e, u_e) \in \mathcal{D} \times \mathcal{U}$ is said to be a *operating point* of the system (2.2) if and only if the following relation holds:

$$f(x_e, u_e) = 0 ,$$

where f is the state function of the system (2.2).

Definition 2.3.2 *Set of operating points*: A Set of operating points of the system (2.2), denoted by $\mathcal{E}[\Sigma(\mathcal{H})]$, is the set of operating points chosen by some rule:

$$\mathcal{E}[\Sigma(\mathcal{H})] := \{(x_e, u_e) : [f(x_e, u_e) = 0] \ , (x_e, u_e) \in \mathcal{D} \times \mathcal{U}\} \ ,$$

where set $\mathcal{E}[\Sigma(\mathcal{H})]$ has the finite number of elements.

Definition 2.3.3 For a given system, a local linear model at the operating point $(x_{ei}, u_{ei}) \in \mathcal{E}[\Sigma(\mathcal{H})]$, denoted $\Sigma(\mathcal{H}_{Li})$, is defined as follows:

$$\Sigma(\mathcal{H}_{Li}) \begin{cases} \dot{\xi}(t) = A_i\xi(t) + B_i\eta(t) \\ \theta(t) = C_i\xi(t) + D_i\eta(t) \end{cases} , \tag{2.4}$$

where i is the scheduling index, ξ, η and θ correspond to small perturbations of x, u and y at the operating point (x_{ei}, u_{ei}):

$$\xi = x - x_e \quad \eta = u - u_e \quad \theta = y - y_e \ .$$

All of the signals ξ, η and θ are elements of a binormed linear space $\Omega \subset \mathcal{L}$:

$$\Omega := \{\sigma \in \mathcal{L} : \|\sigma\|_{\mathcal{L}_\infty} < \infty \wedge \|\sigma\|_{\mathcal{L}_2} < \infty\} .$$

The matrices A_i, B_i, C_i and D_i of linear model are obtained as follows:

$$A_i := \frac{\partial}{\partial x} f(x,u)|_{(x_{ei},u_{ei})} \quad B_i := \frac{\partial}{\partial u} f(x,u)|_{(x_{ei},u_{ei})}$$
$$C_i := \frac{\partial}{\partial x} h(x,u)|_{(x_{ei},u_{ei})} \quad D_i := \frac{\partial}{\partial u} h(x,u)|_{(x_{ei},u_{ei})}$$

at the operating point (x_{ei}, u_{ei}). The bound of signals ξ, η and θ are as follows:

$$\eta \in \mathcal{U}_{\lambda_i} \subset \{\sigma \in \Omega : \|\sigma\|_{\mathcal{L}_\infty} < \lambda_i\}$$

$$\xi \in \mathcal{X}_{\gamma_i} \subset \{\sigma \in \Omega : \|\sigma\|_{\mathcal{L}_\infty} < \gamma_i\}$$

$$\theta \in \mathcal{Y}_{\omega_i} \subset \{\sigma \in \Omega : \|\sigma\|_{\mathcal{L}_\infty} < \omega_i\}$$

Definition 2.3.4 *Indexed set of local linear models*: Is a set of all linear multiple models used for approximation of the nonlinear system model:

$$\Sigma(\mathcal{H}_M) := \{\Sigma(\mathcal{H}_{Li}),\ i = 1..N\}\ .$$

These definitions help us to conduct the proposed design procedure. Linear models for approximating behavior of a nonlinear system at some operating point (x_{ei}, u_{ei}) are defined with the information on boundaries of small perturbed signals.

Chapter 3

Stability Analysis

In this section we summarize several basic stability notions needed in later sections of the report.

3.1 Lyapunov Stability

Consider the autonomous system:

$$\dot{x} = f(x) \tag{3.1}$$

where $f : D \rightarrow R^n$ is a locally Lipschitz function mapping a domain $D \in R^n$ into R^n. An equilibrium point of (3.1) is denoted as $x_e \in D$, and we know that $f(x_e) = 0$. Without

loss of generality we assume that $x_e = 0$.

Definition 3.1.1 *Positive-definite function* centered at $x_0 \in D \subseteq R^n$, a function $V : D \to R$ is a positive-definite function centered at x_e if $V(x_e) = 0$ and $V(x) > 0$ for all $x \neq x_e$. If $x_e = 0$, we just call $V(x)$ a positive definite function.

Definition 3.1.2 *Proper function*: A function is proper if it is continuous, and both the function and its inverse maps a compact set into another compact set and vice versa. A proper positive-definite function $V(x)$ is radially unbounded, i.e.:

$$\lim_{\|x\| \to \infty} V(x) = \infty \, .$$

Definition 3.1.3 *The equilibrium point* $x_e = 0$ of (3.1) is:

- *stable* if, for each $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0 \, .$$

- *unstable* if not stable.
- *asymptotically stable* if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0 \, .$$

Before we state the Lyapunov stability theorem, let us define the derivative of V along the trajectories of (3.1):

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ \ldots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \ldots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x) \ . \tag{3.2}$$

We denote the time derivative of a Lyapunov function V by the product of its partial derivate and the system function.

We now state Theorem 3.1.1 which contains the main results in the Lyapunov theory.

Theorem 3.1.1 Let $x = 0$ be an equilibrium point for (3.1) and $D \in R^n$ be a domain containing $x = 0$. Let $V : D \rightarrow R$ be a positive definite function, such that:

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$

$$\dot{V}(x) \leq 0 \text{ in } D$$

then, $x = 0$ is locally stable. If, in addition,

$$\dot{V}(x) < 0 \text{ in } D - \{0\}$$

then $x = 0$ is locally asymptotically stable.

Relying only on Lyapunov direct method is not sufficient for our research, since it is possible just to treat the autonomous class of systems (3.1). For analysis of forced systems, e.g. (2.2), it is necessary to use more powerful tool such as input-to-state stability concept introduced by Sontag [15], [16], [17].

3.2 Input-Output Stability Theory

We now present the concept of input-output stability for the case of the system defined by expression (2.2).

Definition 3.2.1 A mapping $H : \mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$ is $\mathcal{L}_p$-stable if there exist a class $\mathcal{K}$ function α, defined on $[0, \infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_\mathcal{L} \leq \alpha(\|u_\tau\|_\mathcal{L}) + \beta \tag{3.3}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

It should be noted that in the definition of input-output stability i.e. $\mathcal{L}_p$-stability we do not have notion of *initial state* $x(0)$. As a matter of fact this stability definition is based on the input-output representation of the system and does not contain any information on state variables of the system.

The notion of the finite gain $\mathcal{L}_p$-stability is defined as follows:

Definition 3.2.2 A mapping $H : \mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$ is finite-gain $\mathcal{L}_p$-stable if there exist constants γ and β, both elements of $R^+ + \{0\}$, such that:

$$\|y_\tau\|_{\mathcal{L}_p} \leq \gamma \cdot \|u_\tau\|_{\mathcal{L}_p} + \beta \tag{3.4}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

The following theorem taken from reference [2], (see also e.g. [29]) specifies conditions for $\mathcal{L}_2$-stability. The main importance of the theorem is that it provides a method to determine an upper bound for the $\mathcal{L}_2$ gain of a nonlinear system given in the state space form.

Theorem 3.2.1 The system (3.5) is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ, if for each $x_0 \in R^n$, the following holds:

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \quad x(0) = x_0 \\ y &= h(x) \end{aligned} \tag{3.5}$$

- $f(x)$ is locally Lipschitz, $G(x)$, $h(x)$ are continuous over R^n, where $G(x)$ is a matrix $n \times m$, and $h : R^n \rightarrow R^q$.

- $f(0) = 0$ and $h(0) = 0$.
- There exist $\gamma \in R^+$, and a continuously differentiable, positive semidefinite function $V(x)$ such that the following inequality (Hamilton-Jacobi inequality) is satisfied:

$$\mathcal{H}(V, f, G, h, \gamma) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) (\frac{\partial V}{\partial x})^T + \frac{1}{2} h^T(x) h(x) \leq 0 \,. \quad (3.6)$$

□

The theorem is formulated for the class of input affine systems, a useful and large class of systems.

In the sequel, we will also use Input-Output Gains of LTI[1] systems in $\mathcal{L}_\infty$ sense. For an input-output stable LTI system we have that:

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \gamma_\infty \cdot \|u_\tau\|_{\mathcal{L}_\infty} \quad (3.7)$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

[1]Linear Time Invariant

The above expression is the classical definition of system gain. If the LTI system is represented by its impulse response $h(\cdot)$, then

$$y = \int_0^t h(\tau)u(t-\tau)\,\mathrm{d}\tau \;=\; h(t) * u(t)\,.$$

It is well known that, if $h(\cdot)$ is given by

$$h(t)\begin{cases} h_0\delta(t) + h_a(t), & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where $h_0 \in \mathbb{R}^{q\times 1}$, $\delta(t)$ denotes the unit impulse, and $h_a(\cdot) \in \mathcal{L}_1$, then the $\mathcal{L}_\infty$ norm of the system ψ is given by

$$\gamma_\infty = \|h_0\|_1 + \|h_a(t)\|_{\mathcal{L}_1}$$

In other words, in the special case of strictly proper LTI systems, the $\mathcal{L}$-infinity gain equals the $\mathcal{L}_1$ norm of the impulse response. If the LTI system is not strictly proper, then the impulse response of the system contains impulse functions, and the additional term $\|h_0\|_1$ must be added in the computation of the system gain.

3.3 Input to State Stability

We introduce the notion of input-to-state stability for the case of forced systems:

$$\dot{x} = f(x,u) \tag{3.8}$$

where $f : D \times D_u \rightarrow R^n$ is locally Lipschitz in x and u, $D \subset R^n$ is a domain that contains $x = 0$, and $D_u \subset R^m$ is a domain that contains $u = 0$. The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$.

First, we need to define the classes of functions that we use in the definitions.

Definition 3.3.1 A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Definition 3.3.2 A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

Definition 3.3.3 The system (3.8) is said to be locally input-to-state stable if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ, and positive constant k_1 and k_2 such that for any initial state $x(t_0)$ with $\|x(t_0)\| < k_1$ and any input $u(t)$ with $sup_{t \geq t_0} \|u(t)\| < k_2$, the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(sup_{t_0 \leq \tau \leq t} \|u(\tau)\|) \tag{3.9}$$

for all $t \geq t_0 \geq 0$. it is said to be input-to-state stable if $D = R^n$, $D_u = R^m$, and inequality (3.9) is satisfied for any initial state $x(t_0)$ and any bounded input $u(t)$.

Inequality (3.9) implies explains that for a bounded input $u(t)$, the state vector $x(t)$ will be bounded. The ultimate bound of the state vector $x(t)$ is a class $\mathcal{K}$ function of $sup_{t_0 \leq \tau \leq t} \|u(\tau)\|$. It is easy to show that $x(t)$ converges to zero if $\|u(t)\|$ converges to zero as $t \to \infty$.

The following Theorem provides sufficient conditions for input-to-state stability.

Theorem 3.3.1 The system $f(x, u)$ is locally input-to-state stable if there exist a continuously differentiable function $V : D \to R$ and class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ and χ satisfying:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in D \tag{3.10}$$

$$\frac{\partial V}{\partial x}[f(x,u)] \leq -\alpha_3(\|x\|) + \chi(u), \quad \forall x \in D, u \in D_u \tag{3.11}$$

Moreover if $D = R^n, D_u = R^m$ and $\alpha_1, \alpha_2, \alpha_3$ and χ are class $\mathcal{K}_\infty$, then the system $\dot{x} = f(x, u)$ is input-to-state stable.

Definition 3.3.4 A function $V : D \to R$ that satisfies the conditions of Theorem 3.3.1 will be referred to as an ISS Lyapunov function

In order to prove by Theorem 3.3.1 that is a system input-to-state stable we have to chose a proper function $V(x)$ and a class $\mathcal{K}$ function $\alpha_3(\cdot)$.

For nonlinear systems, the $\mathcal{L}_\infty$ gain of the mapping from input-to-state can be approximated using an argument based on Theorem 3.3.1 and the concept of *ultimate gain*. Notice that, if the conditions of Theorem 3.3.1 are satisfied, then assuming that the input function satisfies a bound of the form $\sup_{t_0 \leq \tau \leq t} \|u(\tau)\| = r_u > 0$, then all trajectories converge to a region of the state space satisfying

$$\alpha_3(\|x\|) \leq \chi(r_u)$$

i.e.

$$\|x\| \leq \alpha_3^{-1}(\chi(r_u))$$

and thus, the state x is enclosed in a region

$$B_d = \{x \in \mathbb{R}^n : \|x\| \leq d = \alpha_3^{-1}(\chi(r_u))\}$$

This means that the trajectories $x(t)$ resulting from an input $u : \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \leq r_u > 0$ will eventually enter the region

$$B_d = \max_{\|x\| \leq d} V(x).$$

Thus, we can define the *ultimate gain* γ_{ISS} as follows:

Definition 3.3.5 The ultimate gain γ_{ISS} in the sense of $\mathcal{L}_\infty$ norm for a input-to-state stable system is:

$$\gamma_{ISS} = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \circ \chi \,. \tag{3.12}$$

For the choice $\alpha_3(x) = V(x)$, where $V(x)$ is a proper function:

$$\gamma_{ISS} = \alpha_1^{-1} \circ \chi \,. \tag{3.13}$$

The latter definition 3.13 will be used throughout the thesis.

3.4 The Concept of Stable Motion

As mentioned in the introduction, one of objectives of this thesis is to address control design for systems described by multiple models. It seems clear that the stability of systems described by multiple linear models cannot be captured using any of classical stability concepts introduced in sections 3.1, 3.3, 3.2. Indeed, *all* of the classical notions of stability refer to a single system model, linear or not, but cannot account for transitions between several models and/or different operating points. In order to cast the design

problem in the context of a proper theory, in this thesis we will consider an extension of the previous stability concepts, referred to as *stable motion.*

The term *stable motion* is used to describe the ability of a nonlinear control system to move from point A to point B in R^n state space without loss of stability at any point of trajectory as it is showed at figure (3.1). Practically it means that the whole trajectory of the system remains within an *extended stability region.*

Let us consider the nonlinear system (2.2). Consider also N operating points $(x_{i,e}, u_{i,e})$, and an *Indexed set of stabilizing linear controllers* $\Sigma(\mathcal{C}_M)$, *i.e.*, all of the models are controllable. Thus, all of the closed loop systems $\Sigma(\mathcal{H}, \mathcal{C}_i)$, $i = 1, .., N$ are input to state stable under assumption that input signal u is always in $\mathcal{D}_{i,u}$ for some i, and the initial value $x_{i,0}$ is in $\mathcal{D}_i$ for the same i. This means that for all of the input to state stable closed loop systems $\Sigma(\mathcal{H}, \mathcal{C}_i)$, $i = 1, .., N$ at corresponding operating point (x_{ei}, u_{ei}) the following holds:

$$u(t) \in \mathcal{D}_{i,u} \ \wedge \ x_{i,0} \in \mathcal{D}_i \Rightarrow x(t) \in \mathcal{D}_i \ , \tag{3.14}$$

Or, in simple words, for bounded input $u(t)$ and the initial state $x_{i,0}$ sufficiently close to the operating point (x_{ei}, u_{ei}) the state $x(t)$ of the input to state stable system $\Sigma(\mathcal{H}, \mathcal{C}_i)$ remains bounded for all times.

Using the definition of *a local linear model* (2.3.3) and the implication (3.14) from

above, we can define *the stable motion* as follows [27]:

Definition 3.4.1 A system experiences *a stable motion* from an initial state x_1 at $t = 0$ to a final state x_N at $t = t_N$ if the following conditions are satisfied:

- There are obtained N locally input to state stable closed loop systems $\Sigma(\mathcal{H}, \mathcal{C}_i)$, $i = 1, .., N$ at the corresponding operating points $(x_{e,i}, u_{e,i})$ with $x_1 \in \mathcal{D}_1$ and $x_N \in \mathcal{D}_N$;
- For each $t = \tilde{t}$ the following holds:
  - $x(\tilde{t}) \in \mathcal{D}_i$ for some $i = 1, .., N$,
  - $\exists \epsilon \; : \; u(t) \in \mathcal{D}_{i,u}$ for $\tilde{t} \leq t \leq \tilde{t} + \epsilon$.

Sufficient conditions for a system to perform *stable motion* are given in the following theorem [27]. We assume that every point in each set $\mathcal{D}_i$ is reachable from x_{ei}

Theorem 3.4.1 The system $\Sigma(\mathcal{H}, \mathcal{C}_M)$ is possible to steer from an initial state $x_1 \in R^n$ to a final state $x_N \in R^n$ at $t = t_N$ following a stable motion if the following conditions are satisfied:

- $x_1 \in \mathcal{D}_i \; \wedge \; x_N \in \mathcal{D}_N$;

- $\mathcal{D}_i \cap \mathcal{D}_{i+1} \neq \emptyset, \;\; i = 1, 2, ..., N-1$.

The proof is omitted here, and it can be found in [27].

The above theorem says that if we have a connected set $\mathcal{D} = \cup_{i=1}^{N} \mathcal{D}_i$, which we call *extended stability region*, obtained for the system $\Sigma(\mathcal{H}, \mathcal{C}_M)$, we may proceed with looking for a control law to steer the system from A to B.

If there exist a control law for $u(t)$ such that $u(t) \in \mathcal{D}_{i,u}$ for any $i \in \{i, .., N\}$ that steers the $\Sigma(\mathcal{H}, \mathcal{C}_M)$ from some point $A \in \mathcal{D}$ to another point $B \in \mathcal{D}$ in such a way that the state vector variable $x(t)$ of $\Sigma(\mathcal{H}, \mathcal{C}_M)$ stays within $\mathcal{D}$ for all t, then it is possible to safely steer the system, and it performs *stable motion* trough $\mathcal{D}$.

Definition 3.4.2 *Region of stability* at the operating point (x_{ei}, u_{ei}) of the locally input to state stable nonlinear system $\Sigma(\mathcal{H}, \mathcal{C}_i)$ is the set:

$$\mathcal{D}_i = \{x \in R^n \mid \|x - x_{e,i}\| < r\} \;\; ,$$

where the input u is bounded in:

$$\mathcal{D}_{i,u} = \{u \mid \|u - u_{i,e}\| < r_{i,u}\} \, ,$$

and where $r_i, r_{i,u} \in R$.

If we assume that there exist a *region of stability* around each of the operating points $(x_{i,e}, u_{i,e}) \in \mathcal{E}[\Sigma(\mathcal{H})]$, we have an *extended region of stability*, which is the union of all *regions of stability* and connected.

Definition 3.4.3 *Extended region of stability* $\mathcal{D}$ is the union of *regions of stability* $\mathcal{D}_i$ that are defined for input to state stable systems $\Sigma(\mathcal{H}, \mathcal{C}_i)$, $i = 1, .., N$ with bounded input signal u around $u_{i,e}$ by $\|u - u_{i,e}\| < r_{i,u}$, $r_{i,u} \in R$,

$$\mathcal{D} = \{\cup_{i=1}^{N} D_i\}$$

such that:

$$\mathcal{D}_i \cap \mathcal{D}_{i+1} \neq \emptyset, \; i = 1, .., N \; .$$

In this thesis the existence of *regions of stability* for a nonlinear control system is tested by investigating the input to state stability at each of the operating points derived for the system. In order to obtain an *extended region of stability* it is necessary to find the region of stability in the vicinity of each operating point and check how they connect among themselves. If neighboring regions overlap as is shown in Figure (3.1), the *extended region of stability* exists and it is possible to steer the system from point A to

point B through the stability regions under the assumption that it is possible to find a control law $u(t)$ that will preserve the stability of the systems within designated regions.

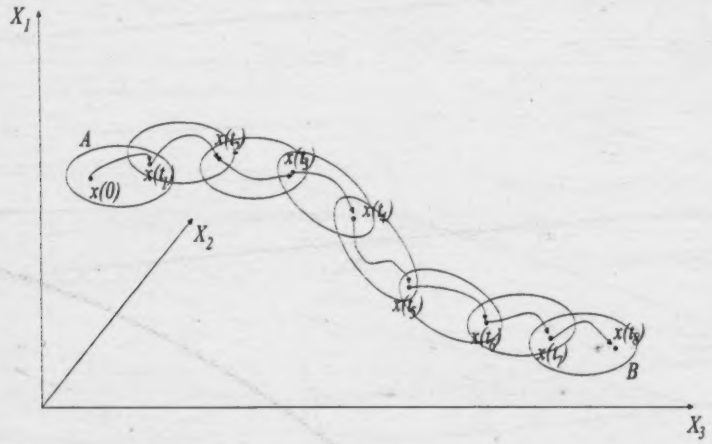

Figure 3.1: Stability regions and the trajectory of a stable motion

To construct the control system that performs stable motion it is necessary to solve the following problems:

1. Select a suitable number of operating points, and design local controllers to operate at each one of them. The number of operating points and their location should be sufficient to satisfy the following criteria:

   - should form a connected region in the state space which covers all points of interest in the particular application considered.

- performance is guaranteed over the operating region of each individual controller, despite of the fact that the controller is designed based on a linear approximation of the plant.

2. Derive a control law that provides safe and stable transitions between operating points.

3.5 The Small-Gain Theorem

The Small-Gain Theorem represents one of the most important results in system theory. It is also famous for its generality and simplicity. The theorem was formulated by Zames [22],[23] and it is quoted by virtually all books dealing with system theory (e.g. [29], [9], [64] etc.). We consider the classical version of The Small-Gain Theorem based on finite-gain $\mathcal{L}_p$ stability definition.

For the system on Fig. (3.2) we assume the following:

1. for any pair of inputs $u_1 \in \mathcal{L}_e^m$ and $u_2 \in \mathcal{L}_e^q$, there exist unique outputs $e_1,\ y_1 \in \mathcal{L}_e^m$ and $e_2,\ y_2 \in \mathcal{L}_e^q$;

2. both systems $H_1 : \mathcal{L}_e^m \to \mathcal{L}_e^q$ and $H_2 : \mathcal{L}_e^q \to \mathcal{L}_e^m$ are finite-gain $\mathcal{L}_p$ stable, *i.e.*:

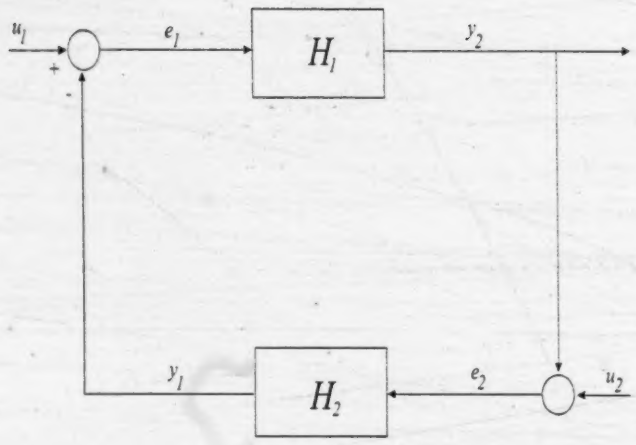

Figure 3.2: Feedback connection

$$\|y_{1\tau}\|_{\mathcal{L}_p} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}_p} + \beta_1, \ \forall e_1 \in \mathcal{L}_e^m, \forall \tau \in [0, \infty) \tag{3.15}$$

$$\|y_{2\tau}\|_{\mathcal{L}_p} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}_p} + \beta_2, \ \forall e_2 \in \mathcal{L}_e^q, \forall \tau \in [0, \infty) \tag{3.16}$$

Theorem 3.5.1 For the feedback system in Fig. (3.2) under assumptions 1 and 2 above, if

$$\gamma_1 \cdot \gamma_2 < 1 \tag{3.17}$$

then for all $u_1 \in \mathcal{L}_e^m$ and $\dot{u}_1 \in \mathcal{L}_e^m$ the following holds:

$$\|e_{1\tau}\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_{1\tau}\|_{\mathcal{L}_p} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}_p} + \beta_2 + \gamma_2\beta_1) \tag{3.18}$$

$$\|e_{2\tau}\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_{2\tau}\|_{\mathcal{L}_p} + \gamma_1\|u_{1\tau}\|_{\mathcal{L}_p} + \beta_1 + \gamma_1\beta_2) \tag{3.19}$$

$\forall\tau \in [0,\infty)$.

In the case when $u_1 \in \mathcal{L}_p^m$ and $u_2 \in \mathcal{L}_p^q$, we have that $e_1,\ y_1 \in \mathcal{L}_p^m$ and $e_2,\ y_2 \in \mathcal{L}_p^q$. Also, the norms of e_1 and e_2 are bounded by the right hand side expressions above.

Chapter 4

Survey of Control Techniques

In this chapter we present a brief survey of existing control design methods. The application range of the methods, their limitations, virtues and drawbacks are also discussed in order to explain the place and role of the Gain Scheduling design method in control practice. A survey of linear and nonlinear control design methods is given in sections 4.1 and 4.2 respectively. Section 4.3 surveys the elements of the linear robust control theory used in this thesis. Various approaches to gain scheduling are discussed in section 4.4, because of their close relationship to this research.

4.1 Linear Control Design Methods

Historically, control system theory was developed for linear systems. Linear models of real physical devices can only approximate the behavior of the system about an operating point. The term of "operating point", which is also called equilibrium point, set point, operating condition [75], trim point [51], [52], [53], equilibrium state [36]), could be defined as: a point on the system's characteristic curve, defined by expression $f(x, u) = 0$, in the vicinity of which the system is working under nominal conditions. Defining the term with the flavor of the nonlinear system analysis, we can say: "a point where the system states can stay forever" [36].

Many techniques are available for designing linear control systems, including model predictive control [69], LQR, LQG, as well as the popular $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal design [39], [41], [73], [42], [12], [63], [40]. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ deal with the synthesis of controller to satisfy a prespecified performance for a particular closed loop transfer function.

4.2 Nonlinear Control Design Methods

Nonlinear design methods could be classified in two major groups: Lyapunov based design and differential geometric design methods.

The main representatives of the first group are backstepping and Lyapunov redesign.

The focus of Lyapunov based methods is the design of an asymptotically stabilizing feedback law. This is accomplished by forcing the time derivative of a so-called Lyapunov function to be negative along the closed loop system trajectories.

The problem could be solved in two ways. First, by finding the control law for $u(x)$ and then checking the stability of the overall closed loop system. Second, by finding a Lyapunov function and then deriving from it a suitable control law in the form of the dynamic system in order to make the Lyapunov function to decrease along the closed-loop solutions. The problem is not simple; indeed, finding a suitable Lyapunov function for the given system could be difficult or even impossible in some cases. The given system practically introduces implicitly a constraint on the Lyapunov function, and the whole procedure depends heavily on the structure of the system. For example, backstepping requires a special systems structure called triangular.

Backstepping is a recursive method based for the construction of the suitable Lyapunov function connected to the design of feedback control [29], [58], [49].

The Lyapunov Redesign method provides the sufficient conditions for stability robustness in any design method [48], [49], [29]. Using Lyapunov Redesign, it is possible to obtain a control system that stabilizes the actual plant in the presence of model uncertainty.

With respect to differential geometric methods here we mention Exact Feedback Linearization, which is one of the fundamental milestones of this approach. Exact Feedback Linearization is based on the idea of nonlinear cancelation to render the overall closed loop system, linear. Certain structural properties are required for such a cancelation to exist. In practice, exact cancelation is almost impossible due to parameter uncertainties, and computational errors [29]. This is a limitation of this method which requires additional stability and robustness analysis.

All of the above mentioned methods are very sensitive to the precision of the parameters of the model. A slight mismatch of model parameters can deteriorate control performance as well as influence stability of the system. Thus, we can conclude that complexity of a system is the limit.

4.3 Linear Robust Control Technique

As it was mentioned before, the basic objective of any control design is stability. However, this goal is not easy to achieve due to mismatches between the real system and its model. The most common type of model for representing a system at an operating point is linear, and as we know, a linear model is valid only within a small area around the operating point. Usually a linear model is accompanied with an uncertainty that captures

the lack of information about the original system. In this case robust control techniques provide tools for design of stabilizing controllers for linear systems with uncertainties.

The notion of uncertainty represents a mismatch between an actual system and its nominal model. There are several ways of representing an uncertainty based on information available about the uncertainty. There are two main approaches for modeling uncertainties. One is based on frequency domain perturbations of the magnitude of uncertainties. The other is based on variations in the parameters of the system transfer function, or the parameters of its representation in state space. The first is called unstructured and the latter is called structured. There are two types of unstructured uncertainties: additive and multiplicative. Unstructured means that we have knowledge only of the bound of the magnitude of the uncertainty at each frequency. In this thesis we will consider only additive uncertainties, Figure (4.1).

The goal of robust control theory is to ensure that stability and performance requirements are satisfied by the true system, not just by the nominal plant model.

The typical form of describing an uncertainty is represented in Fig. (4.1). Knowing the upper norm of the uncertainty, an optimal control problem (4.1) can be formulated using the Small Gain Theorem (3.5):

$$\begin{aligned} &min \|K(I-GK)^{-1}\|_{\mathcal{L}_p} \\ &K \\ &s.t. \\ &K \ \textit{stabilizes} \ G \end{aligned} \tag{4.1}$$

We have to find such a stabilizing controller for the plant in Fig. (4.1) that minimizes the norm of the plant allowing in that way the largest possible value of the norm of the uncertainty Δ.

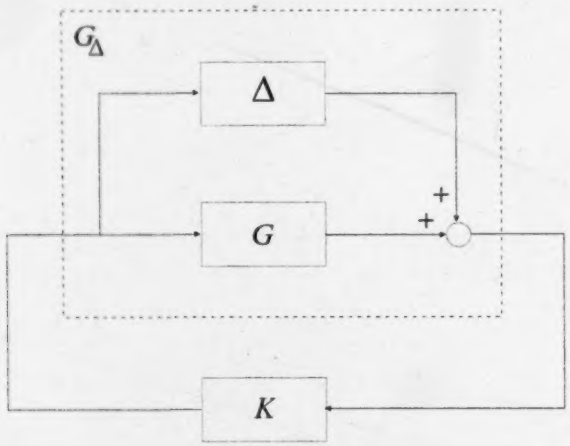

Figure 4.1: Additive Uncertainty

In this thesis, when we obtain $\|\Delta\|_{\mathcal{L}_p}$, instead of solving the mentioned optimization problem we rather chose a controller K such that satisfies the following two conditions:

- K stabilizes G , and

$$\|K(I-GK)^{-1}\|_{\mathcal{L}_p} < \frac{1}{\|\Delta\|_{\mathcal{L}_p}}, \tag{4.2}$$

where the second condition is the consequence of the Small Gain Theorem 3.5.

4.4 Gain Scheduling Control Method

It was explained that linear control techniques are limited by the accuracy of the linear approximation. In practice linear approximations only can produce good results in the vicinity of an equilibrium point. On the other hand nonlinear control techniques are limited by the complexity and the structure of the system. Gain scheduling control methods were introduced in order to fill the gap in between those two streams of control techniques. They have the ability to tackle complex nonlinear control problems by applying linear control techniques over the range of operating points. Due to this ability they have broad scope of applicability.

The general gain scheduled control system is shown in Fig. (4.2). This is a adaptive nonlinear feedback system of a special type; it changes the parameters of the linear regulator as operating conditions change. The parameters of the controller are related

to auxiliary variables that describe the operating conditions. The parameters of the controller are changed without a feedback.

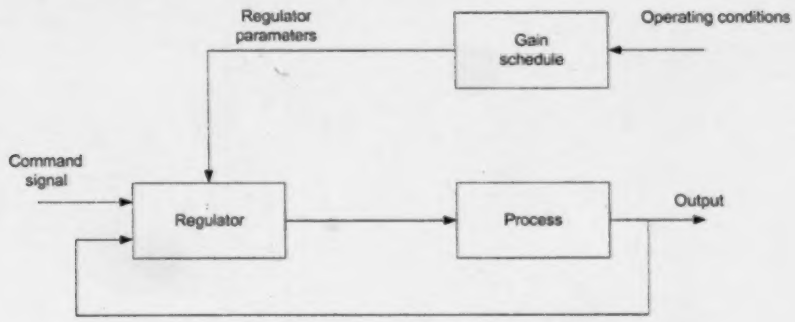

Figure 4.2: Block diagram of the gain scheduling system

The variables used to describe the operating conditions for gain scheduled control depend on the application. Examples are:

1. Scheduling on a prescribed reference trajectory
2. Scheduling on the plant output
3. Scheduling on variables describing environmental conditions
4. Scheduling on state space variables (some or all of them)

and of course, combinations of two or more types. Our choice of the above mentioned scheduling procedure depends on the application and design requirements.

There are no general rules for designing gain scheduling regulators. The key question is to determine the variable that can be used for scheduling the parameters of the controller. There is no precise approach developed to this matter. In practice, engineers are usually guided by common experience such as:

1. The scheduling variable should vary slowly

2. The scheduling variable should capture plant's nonlinearities

It is clear that a gain scheduling variable must represent the operating conditions of the plant. In an ideal case, a simple expression is available to relate the regulated parameters to the scheduling variables. It is necessary to investigate the dynamics of the process in order to apply gain scheduling.

Several methods of the gain scheduling controller design are available, here we will mention three which are the most common:

1. Interpolation methods,

2. Switching methods,

3. Linear Parameter Varying methods.

• Interpolation Methods: Interpolation methods constitute the classical approach to gain scheduling. Traditionally, the development and implementation of gain scheduling

controllers for nonlinear plants involves five steps. Here, we deal with dynamical systems described by equation (2.2).

1. In the first step, it is necessary to investigate the family of operating points of the plant, which is defined by the physical properties of the particular system and the control requirements. It is assumed that system (2.2) is described by a C^1 vector field. It is further assumed that the family of operating points can be parameterized by a vector comprised of chosen scheduling variables.

2. We linearize (2.2) at each element of the family of equilibrium points. The operating points must be sufficiently close to each other in order to approximate the nonlinear model well. In this form we obtain a family of linearized models that is also parameterized by the vector of the scheduling variables. It is important to mention that the set of linear systems at the operating points may be obtained by system identification methods as well.

3. The third step requires designing a family of linear controllers for each model in the family of linear models. The family of linear controllers is thus parameterized by the vector of scheduling variables. The technique used for controller design depends on the requirements and the structure of the family of linear models.

4. The fourth step is the interpolation of the controller parameters whether in state

space form or transfer function form.

5. The fifth step, the last step before the actual implementation, is simulation of the overall system. The point of this step is to check the stability and performance of the overall system. If the design does not comply with the requirements being tested, the designer should return to the first step and do some or all of the following: increase number of operating points, change the linear control technique applied or change the interpolation method.

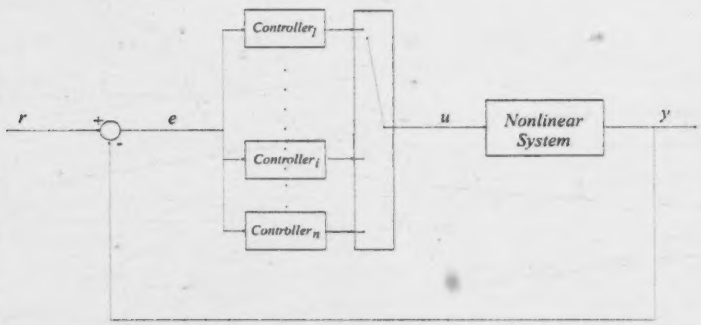

Figure 4.3: Block diagram of a switching gain scheduling control system

• Switching Methods: Their design includes all of above steps except step 4. The structure of a switching system is presented in Fig. (4.3). It is assumed that the nonlinear system under control behaves locally as a linear system. Further it is assumed that the changes of the operating conditions are sufficiently slow. The switching element

from Fig. (4.3) is hooked to the vector of scheduling variables. It is also important to take care of the inactive controllers not to drift away. Thus in the moment of activating a controller the variables should be set to proper initial values.

- Linear Parameter Varying methods (LPV): LPV methods can be conducted in situations where all parameters of state space representation of the family of linear systems can be expressed as homeomorphic functions of the vector of scheduling variables. In this situation all of the scheduling variables must be exogenous.

Common advantages and disadvantages of gain scheduling methods will be described in the following section.

4.4.1 Advantages and Disadvantages

A number of features of the gain scheduling are mentioned below:

Advantages

- Gain scheduling enables wide ranging linear design tools to be employed on difficult nonlinear problems.

• Structural constraints in gain scheduling design do not exist. The only requirement on the state space system model is that the functions that describe it are continuously differentiable in the range of operation.

• It is a big advantage of the gain scheduling that it can be applied in the absence of the analytical plant nonlinear model. The set of models describing the plant may be obtained by identification at the desired operating points.

• In many cases, gain scheduling can be applied intuitively by use of the physical variables in the plant model, without the need for coordinate transformations.

• Since the scheduling variables are selected in order to capture nonlinearities and changes in a plant dynamics as operating conditions change, gain scheduling enables a controller to respond rapidly.

Disadvantages

Of course, the main drawbacks come from the lack of theoretical justification of the method. The steps in gain scheduling technique are *ad-hoc* as it was explained. Thus,

due to the lack of theoretical foundations of stability and performance analysis requires extensive simulations.

Chapter 5

Measures of Nonlinearity

A typical control system design project consists of two steps: (i) find a mathematical model of the plant to be controlled, and (ii) design a controller, based on the model obtained in (i). In the vast majority of cases encountered in industrial applications the dynamical equations describing the plant are nonlinear. For a variety of reasons however, designers often choose to control nonlinear plants by first linearizing and then designing a controller using any of the well developed techniques available for linear time-invariant systems. In order for this approach to be viable, the linear equations obtained after linearization have to be significantly close to the original nonlinear model. If this is not the case, then linear techniques will not lead to satisfactory results.

Nonlinear equations however, come in many forms and deciding when a linear ap-

proximation is "close" to a nonlinear model is not a trivial proposition. The subject has sparked the interest of several researchers who have proposed different "measures" of nonlinearity, in an attempt to determine a limit as to when linear techniques can no longer be theoretically justified and nonlinear controllers must be employed.

In [59] the author defines a special type of two-norm as the vector's inner product of a nonlinear mapping. The measure is easy to obtain through repetition of random input signals with constrained magnitudes. The measure expressed by the newly defined norm was used in [60] to construct a neural network capable of approximating the original nonlinear model.

In [55], [56] and [57], the authors define a measure of nonlinearity based on tangential and normal curvature of a nonlinear system at an operating point. The curvatures are calculated from the first and the second derivative of the model. This measure is used for estimating the interaction effect between the channels of a system in a closed loop. It is also used for transforming a system model in order to improve the controller's performance.

An alternative measure of nonlinearity was introduce by Allgöwer in reference [20]. The focus of this work is on the classification of the type of nonlinearity exhibited by the system. Allgöwer's work was extended by Helbig *et al.* who introduced a computationally efficient lower bound of the nonlinearity measure defined in [20]. See references

[6], [7].

Very recently, references [66], [3], [4], [5] introduce the notion of control-relevant nonlinearity and argue that it is the severity of this effect that dictates the need for nonlinear control. Reference [5] introduces the notion of *optimal control structure* and makes use of nonlinearity measures to determine open-loop system's degree of control-relevant nonlinearity.

Motivated by problems that arise when designing gain scheduling controllers our focus is on the following issue: we consider a nonlinear model and its linearization about an equilibrium point. Considering the fact that the accuracy of the linear approximation deteriorates as the region of operation about the equilibrium point becomes large, we look for a measure of nonlinearity that takes explicit account of the region of operation in state space around the equilibrium point. We propose two new measures of nonlinearity:

(i) First, we view both the nonlinear system and its linearization as mappings from input-to-state. We define the error between the state of the original nonlinear system and that of the linearization and find the region of the state space where this error is bounded in the sense of $\mathcal{L}_\infty$-norm. We cast our solution in the context of the input-to-state theory of systems, and compute the maximum input size that guarantees that the error remains bounded by an arbitrary pre-selected value.

(ii) In our second solution we study the same problem but define our measure of

nonlinearity of the state error in the integral-square (*i.e.* $\mathcal{L}_2$-norm) sense.

Both techniques provide useful information necessary for the implementation of linear multiple models into a gain scheduling design procedure. From the first technique, bounds of input and state signals in $\mathcal{L}_\infty$ norm sense are obtained. From the second technique only the bound on the state signal is obtained. Norms of the error signal, both in $\mathcal{L}_\infty$ and $\mathcal{L}_2$ sense, are used in linear controller design.

5.1 State-Error Norm Problem Definition

We can state the problem to be studied in this chapter. For the nonlinear system $\Sigma(\mathcal{F})$ given by (2.3) we obtain a linear approximation of the form

$$\dot{\xi} = A\xi + Bu.$$

The matrices A and B are such that

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} \quad B = G(x)\Big|_{x=0} \ . \tag{5.1}$$

In the sequel we will assume that A is a Hurwitz matrix (*i.e.* the eigenvalues of A are in the left half plane). This assumption implies that the equilibrium state at the

origin of the unforced nonlinear system $\Sigma(\mathcal{F})$ is exponentially stable, and that $\Sigma(\mathcal{F})$ is locally input-to-state stable.

Under these conditions we define the error associated with the linear approximation of the system $\Sigma(\mathcal{F})$, to be the mapping $u \rightarrow e$, defined by the following dynamical equations:

$$\Sigma(\mathcal{E}) \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} f(x) \\ A\xi \end{bmatrix} + \begin{bmatrix} G(x) \\ B \end{bmatrix} \cdot u \\ e = [I \quad -I] \begin{bmatrix} x \\ \xi \end{bmatrix} = C\,\bar{x} \end{cases} \tag{5.2}$$

where $\bar{x} = [x \quad \xi]^T$, and $C = [I \quad -I]$. This error can be represented as shown in figure Fig. 5.1. With this framework we now propose two alternative measures of nonlinearities (see sections 5.2 and 5.3 for precise definitions):

(i) In our first approach we define the measure of nonlinearity associated with system $\Sigma(\mathcal{F})$ as the $\mathcal{L}_\infty$ gain of the error system (5.2). This measure provides a bound of the maximum instantaneous deviation between the true trajectories of the system $\Sigma(\mathcal{F})$ and those of the linear approximation.

(ii) As an alternative to case (i) we propose a second measure of nonlinearity associated with the system $\Sigma(\mathcal{F})$ as the $\mathcal{L}_2$ gain of the error system (5.2). This measure is very different from that in (i) and provides a bound of the worst-case square-integral

error of the deviations between the true trajectories of the system $\Sigma(\mathcal{F})$ and those of the linear approximation.

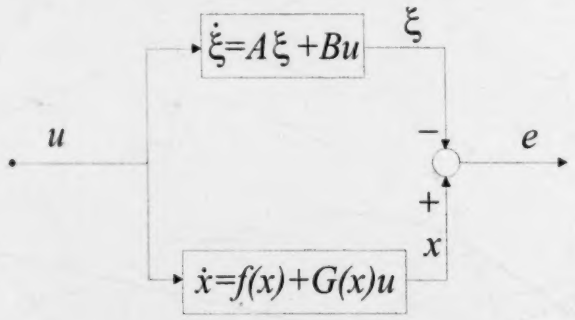

Figure 5.1: The error signal e between states of the nonlinear system and states of its linearization model

5.2 Measuring Nonlinearities in $\mathcal{L}_\infty$-Sense

In this section we elaborate on our first measure of nonlinearity. Throughout this section we will assume that the system $\Sigma(\mathcal{F})$ is input-linear, *i.e.* has the following form:

$$\Sigma(\mathcal{F}) \begin{cases} \dot{x} = f_1(x) + Bu \\ \dot{\xi} = A\xi + Bu \end{cases}, \tag{5.3}$$

We will assume that (5.3) is locally input to state stable, and thus its linear approximation is exponentially stable. Defining the new variable $x = e + \xi$ we can write:

$$\begin{aligned} \dot{\xi} &= A\xi + Bu \\ \dot{e} &= \dot{x} - \dot{\xi} = f_1(e+\xi) - A\xi \,, \end{aligned} \tag{5.4}$$

which can be interpreted as the cascade connection of the linear system $\dot{\xi} = A\xi + Bu$ and the nonlinear system $\dot{e} = f_1(e+\xi) - A\xi$, as shown in Figure (5.2).

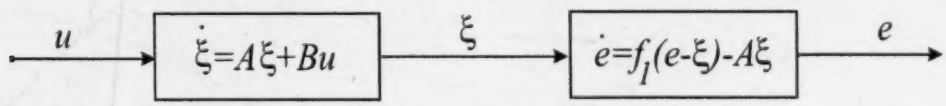

Figure 5.2: Series of two interconnected systems

Employing an ISS analysis for the nonlinear system it is possible to estimate γ_{ISS}, *i.e.* the $\mathcal{L}_\infty$ gain of the mapping from input-to-state. We have:

$$\gamma = \gamma_{\mathcal{L}_\infty} \cdot \gamma_{ISS} \tag{5.5}$$

where $\gamma_{\mathcal{L}_\infty}$ is the $\mathcal{L}_\infty$ gain of the LTI system $\dot{\xi} = A\xi + B\nu$, and γ_{ISS} the ultimate gain of the nonlinear system $\dot{e} = f_1(e+\xi) - A\xi$. The ISS gain γ_{ISS} of the nonlinear subsystem can be estimated based on Theorem (3.3.1). Consider the quadratic Lyapunov function candidate: $V(e) = e^T P e$, where P is obtained from the Lyapunov equation

$$PA + A^T P = -Q \quad Q = Q^T > 0 \tag{5.6}$$

Denoting $\lambda_{min}(P)$ and $\lambda_{max}(P)$ the minimum and maximum eigenvalues of the matrix P and substituting $V = e^T Pe$ in Theorem (3.3.1), we obtain:

$$\lambda_{min}(P)\|e\|^2 \leq V(e) \leq \lambda_{max}(P)\|e\|^2 \tag{5.7}$$

and

$$\frac{\partial V}{\partial e} \times (f_1(e+\xi) - A\xi) \leq -\alpha_3(\|e\|) + \chi(\|\xi\|)\,. \tag{5.8}$$

With this choice of the Lyapunov function, inequality (5.7) is satisfied globally. With respect to the inequality (5.8), the goal is to choose $\alpha_3(\cdot)$ and $\chi(\cdot)$ so that the inequality holds in some region around the origin. Our aim is to estimate the upper bound on the error of the state signals, thus we do not necessarily need to pursue the largest possible region where the system is stable. In this way we have some freedom in choosing $\chi(\cdot)$ and in that way try to obtain lower bound for the ISS gain γ_{ISS}. The value of the gain can be guaranteed over the specified region for an input signal $\|\xi\| < r_\xi$, where r_ξ is the upper bound on the input signal, provided that the initial value of the error is zero $e_0 = 0$. If the upper bound of the norm of the input signal r_ξ is lower, we can obtain lower estimates of the gain. Choosing $\alpha_3(\|e\|_2) = V(e)$, (this is useful choice suggested in [16]) then $\chi(\|\xi\|) = \|\xi\|_2^2$, and we have:

$$\gamma_{ISS}(\|\xi\|_2) = \frac{\|\xi\|_2}{\sqrt{\lambda_{min}(P)}}\,. \tag{5.9}$$

Finally, we have for overall gain:

$$\gamma \leq \gamma_{\mathcal{L}_\infty} \cdot \frac{r_\xi}{\sqrt{\lambda_{min}(P)}} . \tag{5.10}$$

The gain $\gamma_{\mathcal{L}_\infty}$ of the linearized system can be calculated numerically by integrating the absolute value of the impulse response of the system. Thus we have:

$$\|\xi\|_{\mathcal{L}_\infty} \leq \gamma_{\mathcal{L}_\infty} \|u\|_{\mathcal{L}_\infty} , \tag{5.11}$$

and for overall gain γ we can use the following estimation:

$$\gamma \leq \gamma^2_{\mathcal{L}_\infty} \cdot \frac{r_u}{\sqrt{\lambda_{min}(P)}} . \tag{5.12}$$

This estimate of the input to the error gain of a nonlinear system and its linearization can be interpreted as a measure of nonlinearity which depends:

1. on the infinity norm of input signal,

2. on the infinity gain of the linearized subsystem.

This measure is thus useful in the characterization of a nonlinear system behavior around a operating point. It is important to note that we have assumed that the initial

values of state signals of both subsystems are the same $\xi_0 = x_0$, *i.e.* we have assumed that the initial values of the error is zero $e_0 = 0$.

5.3 Measuring Nonlinearities in $\mathcal{L}_2$-Sense

In this section we explore our second measure of nonlinearity. Under the assumptions, the input-affine nonlinear system $\Sigma(\mathcal{F})$ given by (2.3) is locally exponentially stable and knowing that $[f(x)^T \quad (A \cdot \xi)^T]^T$, $[G(x)^T \quad B^T]^T$ and $C \cdot \bar{x}$ of (5.2) satisfy the first two conditions of theorem (3.2.1), we can conclude that there exist $\gamma \in R^+$ and a continuously differentiable, positive semidefinite function $V(\bar{x})$ satisfying inequality (3.6) at least locally within some domain $\bar{\mathcal{D}} \subset R^{2n}$. If the solution $\bar{x}(t)$ of (5.2) does not leave domain $\bar{\mathcal{D}}$, then the error system (5.2) is finite-gain $\mathcal{L}_2$-stable within domain $\bar{\mathcal{D}}$.

Rather than solving the Hamilton Jacobi inequality to estimate a global gain γ associated with the error, our interest is in the following opposite problem: we depart from the assumption that *in a small neighborhood of the origin*, the trajectories of the original nonlinear system $\Sigma(\mathcal{F})$ and those of the linear approximation are indistinguishable. As trajectories deviate further from the origin the error becomes larger. With this background in mind we pursue the following objective:

Given an arbitrarily pre-selected $\mathcal{L}_2$ error-gain γ, small enough so that the linear approximation can be considered as a good model for the original nonlinear system, find (i) the region of the space where this γ is satisfied, and (ii) the bound on the input signals that guarantees that the system trajectories will not leave that region.

An exact solution of these two problems requires solving the Hamilton Jacobi Inequality. Unfortunately. Unfortunately, in general, the Hamilton Jacobi Inequality cannot be solved analytically. Rather than attempting to solve equation, either analytically or numerically, we proceed as follows: we consider a quadratic function of the form

$$V(x) = x^T P x \, , P = P^T \succeq 0 \ . \tag{5.13}$$

With this function, the differential inequality (3.6) reduces to the following:

$$\mathcal{H}(P, f, G, h, \gamma) = 2x^T P f(x) + \frac{2}{\gamma^2} x^T P G(x) G^T(x) P^T x + \frac{1}{2} h^T(x) h(x) \leq 0 \, . \tag{5.14}$$

Hence, the problem reduces to an optimization problem with constraints. We are interested in two following cases:

1. For a given positive $\gamma > 0$ it is necessary to find a symmetric positive definite

matrix P for which the inequality (5.14) holds in the largest possible convex set:

$$\Omega = \{x \in R^n \mid x^T x \leq \varepsilon^2\}\,, \tag{5.15}$$

where the inequality (5.14) is satisfied:

$$\begin{aligned} \varepsilon_{max} &= max\ \varepsilon \\ & P \succeq 0 \\ s.t.\ \ & \mathcal{H}(P, x, \gamma) \leq 0, \quad \forall x \in \Omega(\varepsilon)\,, \end{aligned} \tag{5.16}$$

2. For the given radius ε_0 of Ω, it is necessary to find a symmetric positive definite matrix P for which the inequality (5.14) holds for the minimal positive γ:

$$\begin{aligned} \gamma_{min} &= min\ \gamma \\ & P \succeq 0 \\ s.t.\ \ & \mathcal{H}(P, x, \gamma) \leq 0, \quad \forall x \in \Omega(\varepsilon_0)\,, \end{aligned} \tag{5.17}$$

By formulating the problem in this way, we have to check the second constraint over the whole set $\Omega(\varepsilon_{max})$, which is numerically possible by making a grid over the set. The second constraint can be guaranteed less than zero within Ω if the function in the constraint is *sufficiently smooth* and if the grid is *sufficiently dense.*

The computational burden can be reduced somewhat if the matrix $G(x)$ has a full rank over the set $\Omega - \{0\}$, provided that the matrix P is positive definite rather than positive semidefinite. In this case the following inequality can be derived from (5.14):

$$\frac{1}{\gamma(x, P)} \leq \sqrt{-\frac{4 \cdot x^T P f(x) + h^T(x) h(x)}{4 \cdot x^T P G(x) G^T(x) P^T x}}\,, \tag{5.18}$$

provided that:

$$x^T P G(x) G^T(x) P^T x > 0, \qquad \forall x \in \Omega(\varepsilon_{max}) - 0\,, \tag{5.19}$$

Using the inequality (5.18) problems (5.16) and (5.17) can be reformulated in their simplified forms:

- The problem of estimating the region for a given gain:

$$\begin{aligned} \varepsilon_{max} &= max\ \{\varepsilon = \sqrt{x^T x}\} \\ & P \succeq 0 \\ & s.t. \\ & 4 \cdot x^T P f(x) + h^T(x) h(x) < 0, \qquad \forall x \in \Omega(\varepsilon)\,, \end{aligned} \tag{5.20}$$

- The problem of estimating the gain over a given region:

$$\begin{aligned} \gamma_{min} &= min\ \gamma(x, P) \\ & P \succeq 0 \\ & s.t. \\ & 4 \cdot x^T P f(x) + h^T(x) h(x) < 0, \qquad \forall x \in \Omega(\varepsilon_0)\,. \end{aligned} \tag{5.21}$$

The main obstacle in the solution of the above problems is that the constraints should be tested for very large number of points of the set $\Omega(\varepsilon)$. Once the values of γ_{min} and ε_{max} have been obtained they should be verified by making a dense grid over $\Omega(\varepsilon)$ with

sufficiently fine grid and checking if the constraints are less than zero at each point $x_i \in \Omega(\varepsilon)$. If this test gives negative result, then the same procedure must be repeated taking a larger number of points.

5.3.1 Numerical Calculations

In order to check that the Hamilton-Jacobi inequality (5.14) is satisfied in a region of the state space we propose a numerical search. We define a grid of points in $\mathbb{R}^n$, covering the region of interest, as follows: let $\mathcal{B}$ of orthogonal unit vectors in $\mathbb{R}^n$

$$\mathcal{B} = \{b_1, b_2, b_3, ..., b_n\} \ , \tag{5.22}$$

where $b_i = [0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]^T$ has its only nonzero element in the i-th row. We now construct the set of vectors $\mathcal{Y} \in \mathbb{R}^n$ as linear combinations of the elements on by $\mathcal{B}$ satisfying:

$$\mathcal{Y} = \{y_j = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n, \ j = 1, 2, ..., N_n\} \ , \tag{5.23}$$

where $N_n = m^n$ is the number of vectors in $\mathcal{Y}$ and m is the number of vector in a single plane. The coefficients α_i's are calculated as follows:

$$\begin{aligned} \alpha_1 &= sin(\theta_1) \\ \alpha_i &= \left[\prod_{k=1}^{i-1} cos(\theta_k)\right] \cdot sin(\theta_i) \\ i &= 1, 2, ...n \end{aligned} \tag{5.24}$$

$$\begin{aligned} \theta_j \in \left\{ \left(2\pi \cdot \frac{k}{m}\right),\ k = 0, 1, 2, ..., m; \right. \\ \left. m \in N - \{0\} \right\}\ \ j = 1, 2, ...n. \end{aligned} \tag{5.25}$$

The angles θ_j systematically take values from the set above. The number of grid vectors obtained in this way is $N_n = m^n$. In cases where the function under investigation has certain properties such as being even or odd, it is possible to reduce the number of checking points by four or by two respectively. This can certainly reduce computation time, especially in a higher order problem.

Implementation of Algorithms

As mentioned earlier, it is very difficult to find the "optimal" matrix P that solves problems (5.16) and (5.17). Thus in the case analysis we would use fixed value for matrix P_0. The value is obtained from Lyapunov equation:

$$PA + A^T P = -Q\,,\ Q \succ 0\,, \tag{5.26}$$

or it can be obtained from Riccati equation with some initial guess of gain γ_0:

$$PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C = 0\,, \tag{5.27}$$

The problem (5.16) becomes as follows:

$$\begin{aligned} \varepsilon_{max} &= max\ \varepsilon_1 \\ & P_0 \succeq 0 \\ & s.t.\ \ \mathcal{H}\left(P_0, \left(\varepsilon_1 \cdot \frac{y_j}{\sqrt{y_j^T y_j}}\right), \gamma\right) \leq 0 \\ & \forall \varepsilon_1 \in (0, \varepsilon_{max})\ \ and \quad \forall j \in \{1, 2, ..., N_n\}\,. \end{aligned} \tag{5.28}$$

In the same manner, the problem (5.17) becomes:

$$\begin{aligned} \gamma_{min} &= min\ \gamma_1 \\ & P_0 \succeq 0 \\ & s.t.\ \ \mathcal{H}\left(P_0, \left(\varepsilon_1 \cdot \frac{y_j}{\sqrt{y_j^T y_j}}\right), \gamma_1\right) \leq 0 \\ & \forall \varepsilon_1 \in (0, \varepsilon_{max})\ \ and \quad \forall j \in \{1, 2, ..., N_n\}\,. \end{aligned} \tag{5.29}$$

In above problems, variable ε_1 takes values from interval $(0, \varepsilon_{max})$ with some constant small step that we assume, *e.g.*, $\varepsilon_{step} = 10^{-3}$. This problems are not difficult to solve since we use a constant value for P matrix and check the constraints for all points over the specific set, however they are computationally demanding. In the problem (5.28) for a specified value of γ we increase ε_1 as the constraints are satisfied. For the problem (5.29) we take a value ε_{max} and describe the γ_1 variable as the constraints are satisfied.

Suggested Procedures with P matrix as a variable

If we do not take a constant value for P matrix the situation become more computationally demanding. The main step in each procedure is testing of the constraints. For the whole region Ω we have $N_n \times p$ constraints of the form:

$$\begin{gathered} \mathcal{H}\left(P_0, \left(\varepsilon_1 \cdot \frac{y_j}{\sqrt{y_j^T y_j}}\right), \gamma_1\right) \leq 0 \\ \forall \varepsilon_1 \in (0, \varepsilon_{max}) \ \ and \ \ \ \forall j \in \{1, 2, ..., N_n\}, \end{gathered} \tag{5.30}$$

where p is number of equidistant values of ε_1 in set $(0, \varepsilon_{max})$.

Thus we have $N_n \times p$ inequalities each of them containing elements quadratic and linear in P. The system is over determined since we need only $\frac{n(n+1)}{2}$ linearly independent inequalities for finding a solution of P. For performing procedure it is necessary to solve the set of these $N_n \times p$ inequalities in P:

$$\begin{gathered} \mathcal{H}\left(P, \left(\varepsilon_i \cdot z_j\right), \gamma_1\right) \leq 0 \\ where \ \ z_j = \frac{y_j}{\sqrt{y_j^T y_j}} \\ \varepsilon_i = \frac{\varepsilon_{max}}{m} \times i \ , \ i \in \{1, 2, ..., p\} \ \ and \\ j \in \{1, 2, ..., N_n\} . \end{gathered} \tag{5.31}$$

The following two algorithm can be used in order to solve above defined problems.

Procedure for problem (5.16) is formulated as follows:

1. Take a sufficiently small value for ε_1 and a desired value for γ.

2. Find $P = P^T$ that satisfies the set of $N_n \times p$ inequalities (5.31).

3. if solution for P exists increase ε_1 for a small value $\Delta\varepsilon_1$ and return to the previous step (2).

4. if solution for P does not exist, increase γ or decrease ε_1 and return to the step (2) unless previously achieved ε_1 is satisfactory.

5. end of procedure.

Procedure for problem (5.17) is formulated as follows:

1. Take a sufficiently large value for γ and a desired value for ε_1.

2. Find $P = P^T$ that satisfies the set of $N_n \times p$ inequalities (5.31).

3. if solution for P exists decrease γ for a small value $\Delta\gamma$ and return to the previous step (2).

4. if solution for P does not exist increase γ or decrease ε_1 and return to the step (2) unless previously achieved γ is satisfactory.

5. end of procedure.

Here we will not discuss solving of the set of inequalities (5.31). Due to the number of points that should be taken in account numerical solutions are both time and memory consuming. However, we cannot claim that the above procedures give the best possible solutions.

5.3.2 Upper Bound on Input Signal $u(t)$

For the case of the measure of nonlinearity in the $\mathcal{L}_\infty$ sense we can directly obtain the upper bound on input signal $u(t)$ from (5.12) knowing the limit for the error. For the case of the measure of nonlinearity in the $\mathcal{L}_2$ sense we cannot obtain the bound directly and we have to rely on input-to-state stability analysis as well.

We have found that the gain γ in the $\mathcal{L}_2$ sense is guaranteed over a region $\bar{\mathcal{D}}$ provided that $\|u(t)\|_{\mathcal{L}_\infty} < \delta$ and $\bar{x}$ does not leave the region, i.e $\|\bar{x}\|_{\mathcal{L}_\infty} < \varepsilon_{max}$. To calculate δ we have to know gains in the $\mathcal{L}_\infty$ sense. From the following relations: $e = x - \xi$, $\|e\|_{\mathcal{L}_\infty} \leq \gamma_{ISS}\|\xi\|_{\mathcal{L}_\infty}$, $\|\xi\|_{\mathcal{L}_\infty} \leq \gamma_{\mathcal{L}_\infty}\|u\|_{\mathcal{L}_\infty}$ and $\bar{x}^T\bar{x} \leq \varepsilon_{max}^2$ we derive:

$$\|u\|_{\mathcal{L}_\infty} = \delta \leq \frac{\varepsilon_{max}}{\gamma_{\mathcal{L}_\infty}\sqrt{2(\gamma_{ISS}+1)}} \tag{5.32}$$

Using this magnitude for input signal of the system it is guaranteed that we do not make the error between actual nonlinear states values and values of linearized system states greater than $\delta \cdot \gamma$. Also, norm of states of nonlinear system is bounded as:

$$\|x\|_{\mathcal{L}_\infty} \leq \gamma_{\mathcal{L}_\infty}(1 + \gamma_{ISS}) \cdot \|u\|_{\mathcal{L}_\infty} . \tag{5.33}$$

Thus, we are certain that γ is guaranteed over the actual region:

$$\mathcal{D} = \{x \in R^n \mid \|x\|_{\mathcal{L}_\infty} \leq \delta \cdot \gamma_{\mathcal{L}_\infty}(1 + \gamma_{ISS})\} \tag{5.34}$$

for the nonlinear system.

Chapter 6

Measuring the Nonlinearity of a DC Motor: A Case Study

In this chapter we consider a DC motor and calculate two measures of nonlinearity introduced in the previous chapter at various equilibrium points.

6.1 DC Motor Model

The idea of armature voltage control can be understood looking at the angular velocity equation:

$$\omega = \frac{V_t - I_a R_a}{K_a \phi} \tag{6.1}$$

where V_t is armature voltage, I_a is armature current, R_a is resistance, ϕ is field flux per pole, K_a is armature constant and ω is angular velocity. The armature resistance R_a and the field current I_f are kept constant, and the speed is controlled by varying the armature terminal voltage V_t. In this way the speed is easily controlled from zero to maximum speed in either direction.

The equation relating the variables of series DC motor are:

$$V_t = E_a + I_a(R_a + R_s) + (L_a + L_s)\frac{dI_a}{dt} \tag{6.2}$$

$$J\frac{d\omega}{dt} = T_e - T_{load} - T_{loss} \tag{6.3}$$

where J is the rotor inertia, T_e is air-gap torque, T_{load} is the output or load torque, T_{loss} is the total torque loss by friction and windage, R_a is armature resistance, R_s is stator resistance, L_a is armature inductance, L_s is stator inductance and E_a is back e.m.f. voltage. The back e.m.f voltage E_a and the air gap torque T_e are dependent on the armature current:

$$E_a = K I_a \omega \tag{6.4}$$

$$T_e = K I_a^2 \tag{6.5}$$

where K is a constant. We see from (6.4) that the DC motor inherently possesses negative feedback, *i.e.* an aspect of natural regulation. When the voltage V_t increases the rapid jump of current is constrained by the rise of e.m.f. E_a. Substituting equations (6.5) and (6.4) into (6.2) and (6.3), we obtain:

$$V_t = K I_a \omega + I_a(R_a + R_s) + (L_a + L_s)\frac{dI_a}{dt} \tag{6.6}$$

$$J\frac{d\omega}{dt} = K I_a^2 - T_{load} - T_{loss} \tag{6.7}$$

Using V_t as the control input and the speed ω as the control output in the derivation of the state space model for the series DC motor. The model has two state variables; namely, armature current I_a and angular velocity ω:

$$u = V_t$$

$$x_1 = I_a$$

$$x_2 = \omega$$

$$\dot{x}_1 = -\frac{R_a + R_s}{L_a + L_s} x_1 - \frac{K}{L_a + L_s} x_1 x_2 + \frac{1}{L_a + L_s} u \tag{6.8}$$

$$\dot{x}_2 = \frac{K}{J} x_1^2 - \frac{T_{load} + T_{loss}}{J} \tag{6.9}$$

The motor constant $K(I_a)$ is a nonlinear function of the armature current. Also both load torque $T_{load}(\omega)$ and loss torque $T_{loss}(\omega)$ are nonlinear functions of the angular velocity. Those relations and other constant parameters of the motor can be obtained by experiment. In this report we use the model parameters experimentally obtained in [80]:

$$\dot{x}_1 = a_1 x_1 + a_2 x_1^2 x_2 + a_3 x_1 x_2 + a_4 x_2 + a_5 u \tag{6.10}$$

$$\dot{x}_2 = b_1 x_1^3 + b_2 x_1^2 + b_3 x_1 + b_4 x_2^4 + b_5 x_2^3 + b_6 x_2^2 + b_7 x_2 + b_8 \tag{6.11}$$

where

$$a_1 = -87.5, a_2 = 0.16505031, a_3 = -2.1114092, a_4 = -0.343025, a_5 = 8.33$$

$$b_1 = -3.194522081, b_2 = 40.86598372, b_3 = 6.639199794, b_4 = 1.1838871 \cdot 10^{-5}$$

$$b_5 = -3.937742 \cdot 10^{-3}, b_6 = 0.451613, b_7 = -19.32258, b_8 = -20.71$$

6.2 Measuring Nonlinearity in $\mathcal{L}_\infty$ sense

We now apply to the DC motor equations (6.10) and (6.11) the procedure for measuring nonlinearity in $\mathcal{L}_\infty$ sense. Straightforward computations show that setting $u = u_e =$ 26.10 volts, the system has an (asymptotically stable) equilibrium point at:

$$x_e = \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} 2.07 \\ 9.0 \end{bmatrix}$$

We now study inequality (5.8) in a region about this equilibrium point. Notice that this condition can be rewritten as follows:

$$\frac{\partial V}{\partial e} \times (f_1(e+\xi) - A\xi) + \alpha_3(\|e\|_2) - \chi(\|\xi\|_2) \leq 0 \tag{6.12}$$

Assuming $\|u\|_{\mathcal{L}_\infty} = 0.2$, and using the fact that, at this equilibrium point the gain γ_∞ of the resulting linear approximation is $\gamma_\infty = 0.6455$, we have that $\|\xi\|_{\mathcal{L}_\infty} \leq 0.1291$. Rewriting inequality (5.8) as Figure (6.1) shows an schematic of the several region of the plane that arise in the analysis of inequality (6.12). In the Figure we can distinguish following regions Ω_1, Ω_2, Ω_3, Ω_4, where:

1. Ω_1: In this region the left hand-side of inequality (6.12) is greater than zero, and thus inequality (5.8) is *not* satisfied. In the absence of a forcing force, all trajectories initiating inside this region and close to the equilibrium point converge towards the equilibrium as $t \to \infty$.

2. Ω_2: In this region the left hand-side of inequality (6.12) is less than zero, and thus inequality (5.8) *is* satisfied. It then follows that trajectories initiating in Ω_2 will eventually cross the border of this region and enter Ω_1. Hence, trajectories initiating inside the region Ω_1 cannot leave this region.

3. Ω_3: This region is similar to region Ω_2 in that left hand-side of inequality (6.12) is less than zero, and thus inequality (5.8) is also satisfied. The distinction between

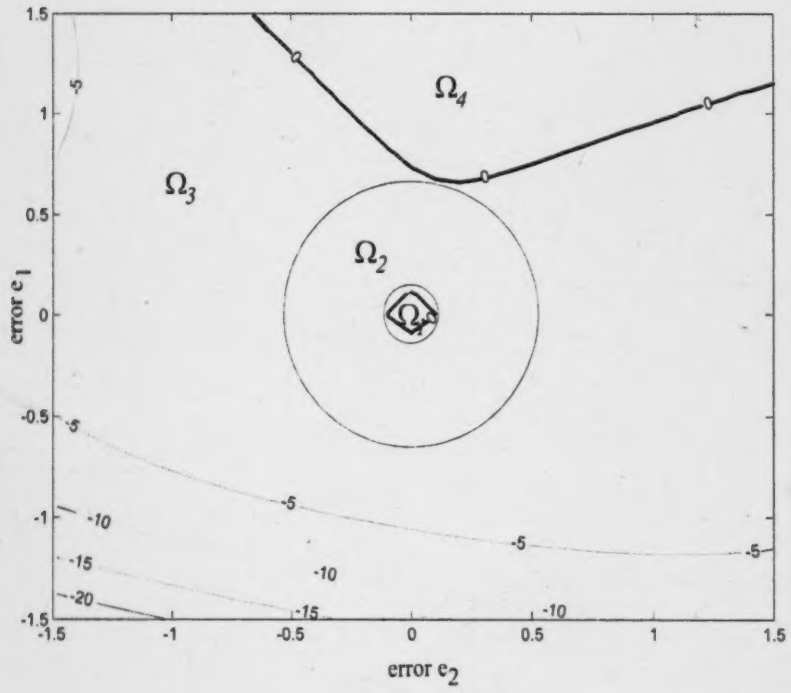

Figure 6.1: Stability region in the error state space

regions Ω_2 and Ω_3 is simply that Region Ω_2 contains the maximum circular region within Ω_3, not overlapping with Ω_4.

4. Ω_4 is the region where inequality (5.8) is greater than zero. Thus, we cannot make any conclusion about stability in this region.

Our goal is to chose an ISS Lyapunov function $V(\cdot)$ and a function $\chi(\cdot)$ to obtain the smallest possible region Ω_1 and the largest possible region Ω_2. To this end we define $V(e) = e^T P e$, where P is obtained from Lyapunov equation (5.6) with $Q = 3I$. The overall gain γ can be calculated graphically from Figure (6.1), by taking the value of the radius r_{Ω_1} of region Ω_1 and dividing it by the infinity norm of the input signal r_u, *i.e.*

$$\gamma = \gamma_G = \frac{\|e\|_{\mathcal{L}_\infty}}{\|u\|_{\mathcal{L}_\infty}} = 0.6 \tag{6.13}$$

where we have denoted this gain by γ_G to indicate that this is the gain computer using the graph of Figure (6.1). This gain can also be obtain using expression (5.10). Computing the gain of the linear approximation, we obtain $\gamma_{\mathcal{L}_\infty} = 0.6455$. Substituting $\gamma_{\mathcal{L}_\infty} = 0.6455$ and with the assuming bound $\|u\|_\infty = 0.2$ we obtain $\gamma = \gamma_F = 0.7$. Thus we conclude that expression (5.10) gives a relatively good estimate of $\mathcal{L}$-infinity gain from the input to the error signal, although not as sharp as the graphical analysis. Expression (5.10) is, however, preferred for systems of large order.

In a typical problem of gain scheduling, the same analysis needs to be performed at various equilibrium points. Table (6.14) represent the gain of the system (5.3) obtained at several operating points, calculated using expression (5.10) and the second row represents the gain obtained by using the graphical procedure described above. Each equilibrium point was obtained by applying a constant input $u = u_{ei}$. The overall gain obtained by calculation is denoted by γ_F and the overall gain obtained from the plot is denoted by γ_G. Operating points are denoted just by its unique value in $\frac{rad}{s}$. The norm of the input to the system $\bar{u} = u - u_{ei}$ was constant $\|\bar{u}\|_{\mathcal{L}_\infty} = 0.2$.

o.p.	1	2	3	4	5	6	7	8
x_2(rad/sec)	9	11	13	15	17	19	21	23
γ_F	0.700	0.824	0.957	1.104	1.269	1.455	1.668	1.911
γ_G	0.60	0.62	0.65	0.7	0.76	0.85	0.95	1.1

(6.14)

From Table (6.14) it can be observed that for the operating points from 1 to 5 equation (5.10) gives a good approximation of the actual gain. As the angular velocity x_2 increases, the approximation deteriorates. Improved results can be obtained changing the ISS Lyapunov function.

Based on the table from Appendix A figure (6.2) was obtained. It can be observed from Fig (6.2) that the estimated gain obtained by graphical method is more accurate than that obtained using equation (5.10). The estimated gain γ_F is growing linearly with respect to the bound on input signal r_u which is a consequence of the procedure

that we performed in order to obtain the gain expression. The estimated gain γ_F is greater than the actual one is due to the majorization that is made for the Lyapunov function using expression (5.7). Note that plots would be the same for γ_F, *i.e.* linear, with respect to the $\|\xi\|_{\mathcal{L}_\infty}$. This is because of the fact that $\|\xi\|_{\mathcal{L}_\infty}$ grows linearly with r_u.

6.3 Measuring Nonlinearity in $\mathcal{L}_2$ sense

For a non-optimal value of P obtained from the Lyapunov equation (6.16) the values of the estimated gain are presented in table (6.15). It is observed that the maximal radius is $\varepsilon_1 = 0.65$ and the estimated gain of the error signal is growing relatively fast. This is illustrated at Fig. (6.3). For values of ε_1 greater than 0.65 it was not possible to prove stability of the system, hence the gain could not be calculated.

In the case where P is obtained from the Riccati equation using $\gamma_0 = 1$ (6.17) we obtain that $\gamma \approx 0$ for an area around origin within radius of $\varepsilon = 0.65$. We cannot make comments on the growth of the gain over this region, but we know that the system is stable within the region.

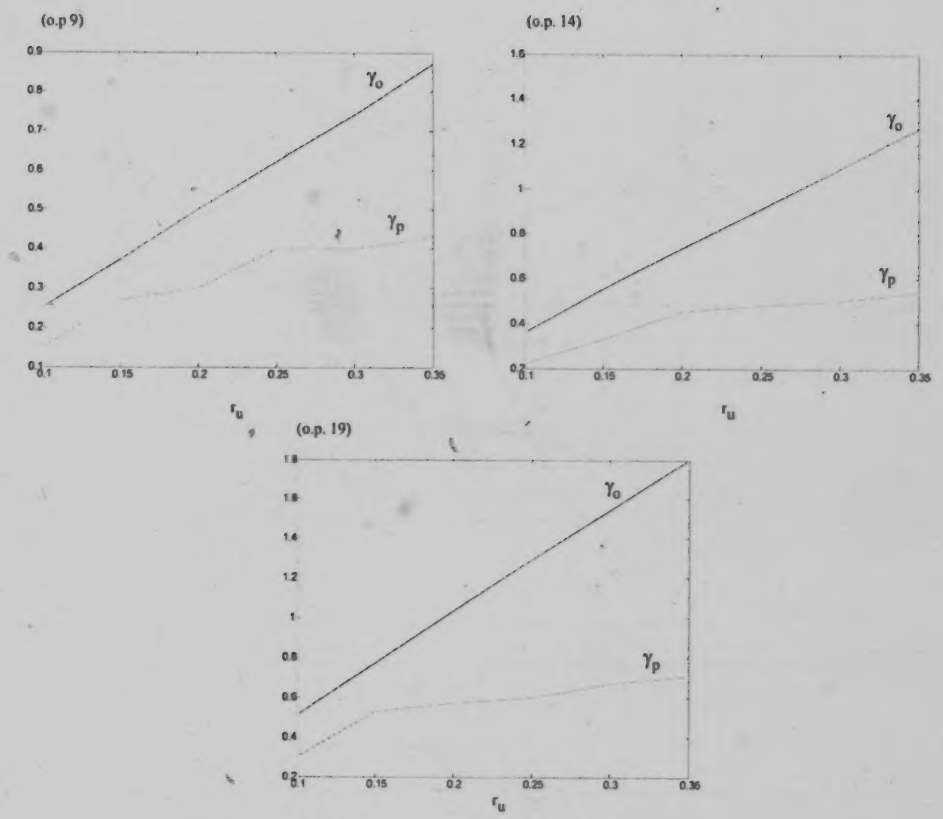

Figure 6.2: Gains' values in the function of the bound of input signal u for 9, 14, 19, $\frac{rad}{s}$

ε_1	0.01	0.1	0.2	0.3	0.4	0.5
γ_{max}	0.12665	0.12791	0.13139	0.13690	0.14686	0.17061
ε_1	0.6	0.65	0.7	0.8	0.9	1
γ_{max}	0.23978	0.35393	*N.A.*	*N.A.*	*N.A.*	*N.A.*

(6.15)

$$P = \begin{bmatrix} 0.0089 & 0.0024 & 0 & 0 \\ 0.0024 & 0.0369 & 0 & 0 \\ 0 & 0 & 0.0089 & 0.0024 \\ 0 & 0 & 0.0024 & 0.0369 \end{bmatrix} \tag{6.16}$$

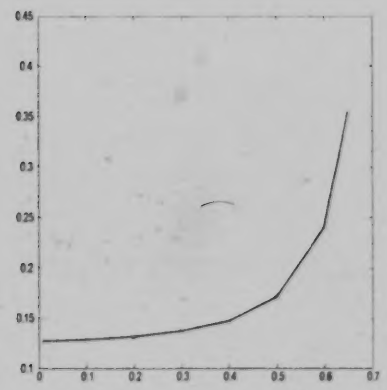

Figure 6.3: The gain γ with respect to the radius ε_1 for DC motor case

$$P = \begin{bmatrix} 0.0059 & 0.0016 & -0.0059 & -0.0016 \\ 0.0016 & 0.0246 & -0.0016 & -0.0246 \\ -0.0059 & -0.0016 & 0.0059 & 0.0016 \\ -0.0016 & -0.0246 & 0.0016 & 0.0246 \end{bmatrix} \tag{6.17}$$

Chapter 7

Gain Scheduling based on Measures of Nonlinearity

It was explained that measures of nonlinearity defined in this work represent the norm of the additive uncertainty of a linear multiple model (see Fig. (4.1)). Using these measures in this sense we can easily apply some of the classical robust control techniques as it was explained in section 4.3. This represents a basic element toward using measures of nonlinearity as an useful tool for design of gain scheduling controllers. In addition to this they define the constraints on state and input signals such that performances of a system are of desired values.

The nonlinearity measure in $\mathcal{L}_\infty$ sense provides us with the upper $\mathcal{L}_\infty$ norm on

uncertainty, the radius of region r_x where the norm is valid and the norm on input signal r_u that preserve ISS stability of the system.

The nonlinearity measure in $\mathcal{L}_2$ sense provides us with the upper $\mathcal{L}_2$ norm on uncertainty and the region r_x where this norm is valid *i.e.* where the system is $\mathcal{L}_2$ stable. However, the upper bound on input signal r_u we can not obtain by this technique. Here, the previous ISS stability analysis provide us with r_u.

Let us describe a gain scheduling closed loop system by using schematic at Fig (7.1). It is an output feedback controller and we describe here its elements: $s = F(x)$, *i.e.* - the switching function, and K - the set of controllers. The idea is that the switching function "lights up" one of the controllers when the system is in corresponding area. We can do that by defining the signal s as a positive integer assigned to each of the areas. This signal s is the index of the indexed set of controllers K. We can write as follows:

$$F(x) = max_{i=1,2,..,Q} \left\{ i \times \frac{sgn\left(\|x - x_i\|_2 \leq r_i\right) + 1}{2} \right\} , \tag{7.1}$$

Where r_i represents the radius of stability region $\mathcal{D}_i$, *i.e.* the region where the norm of the uncertainty is preserved under desired value. For this purpose we take for r_i radius r_x obtained by one of the techniques for nonlinearity measures that we decided to use for controller design. Function $sgn(\cdot)$ is defined as follows:

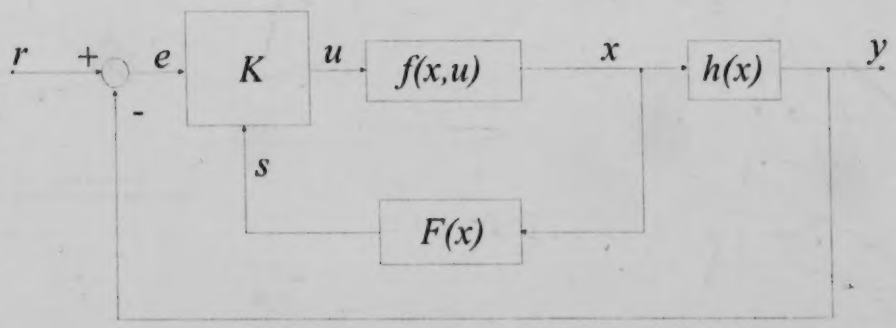

Figure 7.1: Gain Scheduling controller schematic

$$sgn(x) = \begin{cases} 1 & if \quad x \geq 0 \\ -1 & if \quad x < 0 \end{cases} \tag{7.2}$$

For the indexed set of controllers K we have the following:

$$K = \begin{cases} \{\ y = C_1 e + u_1,\ x(0) = x_{0,1} & for\ \ s = 1 \\ \qquad\qquad \vdots & \qquad\quad \vdots \\ \{\ y = C_i e + u_i,\ \ , x(0) = x_{0,i} & for\ \ s = i \\ \qquad\qquad \vdots & \qquad\quad \vdots \\ \{\ y = C_Q e + u_Q\ \ , x(0) = x_{0,Q} & for\ \ s = Q \end{cases} \tag{7.3}$$

In the case where dynamic controllers are employed each of controllers has to be initiated in that way that the control signal $u(t)$ is continuous function of time. The controllers $y = C_i e$ (whether static or dynamic) are designed by some of the linear techniques available at each relevant operating point for each multiple model of the nonlinear system. The output of each controller is added to the value u_i of corresponding

operating point. We assume that the reference signal is not faster then the slowest linear multiple model in the closed loop in order to avoid mismatch in switching. If this is not the case in reality, we can utilize a rate limiter before the closed loop system.

The number of controllers in K and their allocation, *i.e.* the choice of operating points in the characteristic of the model, depend on the requirement for two neighboring regions $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ to be overlapped for all $i \in \{1, 2, .., Q-1\}$, where Q is the required number of operating points. Since regions $\mathcal{D}_i$ are specified as circles by the techniques for calculating measures of nonlinearity, the condition that we have to follow when we chose operating points is:

$$\begin{array}{c} |x_{0,i+1} - x_{0,i}| < r_{x,i+1} + r_{x,i} \ , \\ \forall i \in \{1, 2, .., Q-1\} \ , \end{array} \tag{7.4}$$

where $x_{0,i}$ is the operating point in the state space, and $r_{x,i}$ is the radius of the corresponding region $\mathcal{D}_i$. In this way all neighboring regions are connected, *i.e.* we obtain an extended region of stability:

$$\mathcal{D} = \left\{ \cup_{i=1}^{N} D_i \right\} ,$$

throughout which the system can move without making the error greater than the norm of uncertainty whether in $\mathcal{L}_\infty$ or $\mathcal{L}_2$ sense.

In the following section we will describe how to design the set K regarding which uncertainty norm, *i.e.* nonlinearity measure, we would like to preserve. The both procedure are based on application of the Small Gain Theorem (3.5).

7.1 Application of Nonlinearity Measure in $\mathcal{L}_\infty$ sense

In this section we use our nonlinearity measure in $\mathcal{L}_\infty$ sense in control design using multiple models. We will design the control law empirically such that the Small Gain Theorem is satisfied for additive uncertainty expressed as $\mathcal{L}_\infty$ gain. The radius:

$$r_{x,i} \in \{r_{x,1}, r_{x,2}, ..., r_{x,Q}\}$$

in the switching function (7.1) will be calculated by employing equation (5.33) for each of linear multiple models.

In order to satisfy conditions (4.3) and (4.3) we have to derive the set of controllers such that the following condition is true:

$$\|C_i K_i (I + P_i C_i K_i)^{-1}\|_{\mathcal{L}_\infty} < \frac{1}{\gamma_{\mathcal{L}_{\infty,i}}}, \tag{7.5}$$

for all $i \in \{1, 2, ..., Q\}$. Where $P_i C_i(s)$ is the transfer function of the i^{th} linear multiple model:

$$P_i C_i(s) = C_i(Is - A_i)^{-1} B_i \quad \forall i \in \{1, 2, ..., Q\} \; . \tag{7.6}$$

The $\mathcal{L}_\infty$ norm of the above expression can be calculated as it was explained in section 3.2 and Appendix B.

The set of controllers is represented below:

$$K = \begin{cases} \{y = K_1 e + u_1, & for \;\; s = 1 \\ \quad \vdots & \quad \vdots \\ \{y = K_i e + u_i, & for \;\; s = i \\ \quad \vdots & \quad \vdots \\ \{y = K_Q e + u_Q, & for \;\; s = Q \end{cases} \tag{7.7}$$

Since the controllers are static, we do not need to calculate initial conditions for them.

The set of radia is calculated by equation (5.33) or by observing the plot of equation (5.8). Having all elements calculated we can form function $F(x)$ and the set of controllers K.

7.2 Application of Nonlinearity Measure in $\mathcal{L}_2$ sense

For this case, when additive uncertainty is modeled as the gain $\gamma_{\mathcal{L}_2}$ (upper bound on $\mathcal{L}_2$ norm) from the input signal to the error signal of the system (5.2) we have to satisfy the condition of the Small Gain Theorem in the following form:

$$\|C_iK_i(I + P_iC_iK_i)^{-1}\|_\infty < \frac{1}{\gamma_{\mathcal{L}_2,i}} \ , \tag{7.8}$$

where $P_iC_i(s)$ is the transfer function of the i^{th} linear multiple model. For the indexed set of controllers K we have the following:

$$K = \begin{cases} \begin{cases} \dot{x} = A_1x + B_1e \ , x(0) = x_{0,1} \\ y = C_1x + D_1e \ + u_1 \end{cases} & for \ \ s = 1 \\ \vdots & \vdots \\ \begin{cases} \dot{x} = A_ix + B_ie \ , x(0) = x_{0,i} \\ y = C_ix + D_ie \ + u_i \end{cases} & for \ \ s = i \\ \vdots & \vdots \\ \begin{cases} \dot{x} = A_Qx + B_Qe \ , x(0) = x_{0,Q} \\ y = C_Qx + D_Qe \ + u_Q \end{cases} & for \ \ s = Q \end{cases} \tag{7.9}$$

The regions are defined by the set of radia r_i where a multiple model is valid. A radius is obtained by applying the procedure for nonlinearity measure in $\mathcal{L}_2$ sense for each of operating points. Using this set we define the switching function (7.1).

In the case where dynamic controllers are used the set of initial conditions should be defined in a way that smooth transitions are achieved, *i.e.*, without glitches. We can do that by the following simple rule applied in the moment of switching from i to $i+1$:

$$y(t_s + \epsilon) + u_{(i+1)} = u(t_s - \epsilon) \ , \tag{7.10}$$

thus we have for an initial state the following rule for calculation:

$$\begin{aligned} y(t_s+\epsilon) &= u(t_s-\epsilon) - u_{(i+1)} \\ y(t_s+\epsilon) &= C_i x_{0,i} + D_i e(t_s-\epsilon) \end{aligned}, \tag{7.11}$$

where ϵ is Infinitesimally small time interval and t_s is moment of switching. It is obvious that $x_{0,i}$ of controller is not unique. This is justified by the fact that we take care of the state space values of the nonlinear system. Since controllers are internally stable and linear, regardless of their initial values we do not affect stability of the controller. Thus, the overall closed loop system remain stable in a moment of switching.

7.3 Back to The DC Motor Example

Here we illustrate the gain scheduling controller design for the DC motor model using four operating points. We have used the measure of nonlinearity in $\mathcal{L}_\infty$ sense. For this case we close the loop using proportional gain controllers chosen in order to satisfied condition (7.5) at each of operating point. The procedure described in section 7.1 is applied and results are summarized in the steps below.

1. Operating points are calculated over the region of interest:

i	u_e	x_{1e}	x_{2e}
1	33.2110	2.4398	14
2	34.4859	2.4969	15
3	35.7180	2.5495	16
4	36.9091	2.5978	17

(7.12)

where u_e is expressed in Volts, x_1e in Amperes, and x_2e in radians per second.

2. The following parameters are calculated at each operating point for input variations of $\bar{u} = 0.2$:

o.p.	1	2	3	4
$\gamma_{\mathcal{L}_\infty F}$	1.0293	1.1048	1.1847	1.2695
$\gamma_{\mathcal{L}_\infty G}$	0.7500	0.7500	0.8000	1.0000
r_x	0.5000	0.6000	0.6000	0.5000

(7.13)

The gain from input to the error signal obtained by (5.12) is denoted by $\gamma_{\mathcal{L}_\infty F}$, the gain obtained by graphical method is denoted by $\gamma_{\mathcal{L}_\infty G}$, and the radius where gains are valid is denoted by r_x.

3. The values for K_i are chosen such that expression (7.5) is satisfied. We were calculating the gain of the system as it is explained in Appendix B, for different values of K_i until we obtain a closed loop gain less than $\frac{1}{\gamma_{\mathcal{L}_\infty}}$ from the table above. The calculation results are shown in the following table:

i	1	2	3	4
K_i	0.5	0.5	0.4	0.3
$\gamma_{\mathcal{L}_\infty}$	0.639	0.640	0.499	0.361

(7.14)

We use these parameters for design of the Gain Scheduling controller of the DC motor. The simulation results are presented in figures (7.2)- (7.5). In the first simulation a stare signal was used as the reference signal at the input of the system, Fig. (7.2). The response is presented at Fig. (7.3). It can be observed that the output follows the reference signal fairly well.

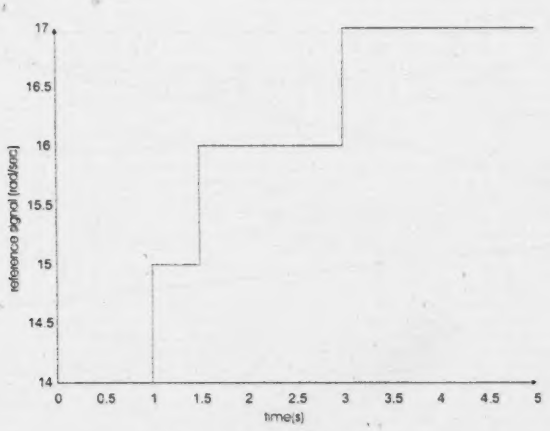

Figure 7.2: The reference input of the Gain Scheduled system

The process of switching is well described by figures (7.4) and (7.5). It can be observed in Fig. (7.4) that functions in expression (7.1) take there values zero or

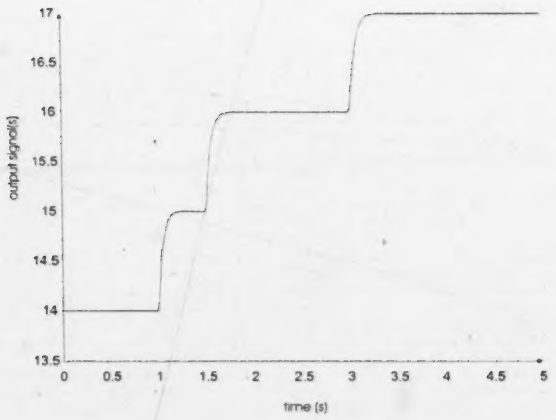

Figure 7.3: The response of the Gain Scheduled system

$i \in \{1, 2, 3, 4\}$ as the output rises, Fig (7.3). Here it was sufficient to consider only the value of the output signal in order to know in which region the system is. It is interesting to see at Fig. (7.4) that two functions from expression (7.1) are active at the same moment when a switching occurs.

The maximum value of expression (7.1) is presented in Fig. (7.5), this is the envelop of the functions in Fig. (7.4). It has the value $i \in \{1, 2, 3, 4\}$ and represents the controller index and the region where system is situated at a particular moment in time.

It is interesting to test the system for signals of different rising time as it was shown in Fig. (7.6). The corresponding response signals are represented in Fig. (7.7). Moments

when the controller is changed are easily recognized in the figure.

It is also interesting to observe the way functions in the switch change their values for different input signals, Fig. (7.8). For the faster input signal, functions stay at values different than zero for a shorter time period. The overlaps of two activated functions at the moments of switching can be observed in all of the plots. This is why the function (7.1) has to have $max(\cdot)$ function in it.

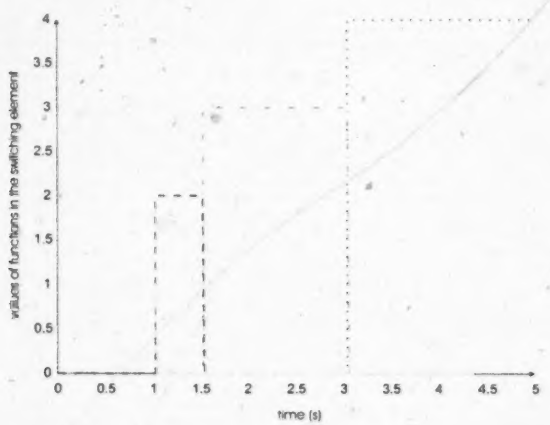

Figure 7.4: Values of functions from (7.1) for the Gain Scheduled system

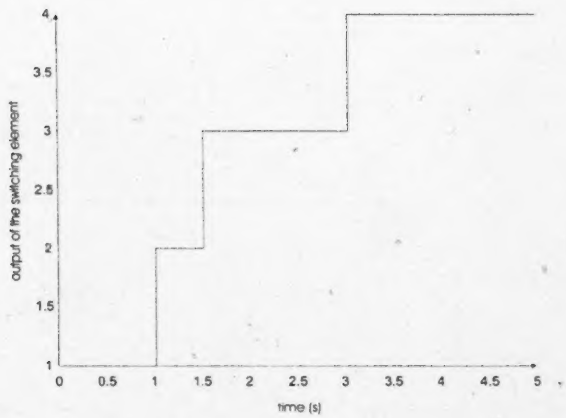

Figure 7.5: Values of index for the Gain Scheduled system

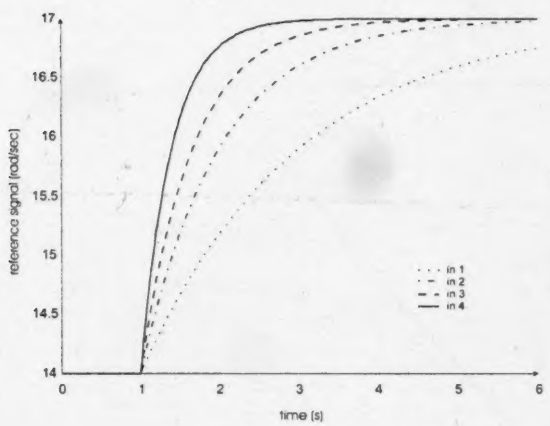

Figure 7.6: Signals of different rising time as the reference signal

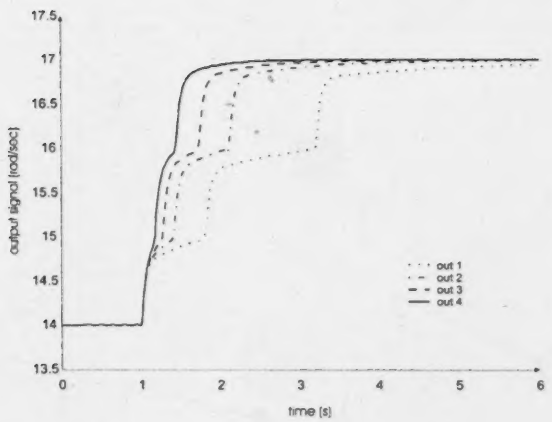

Figure 7.7: System response for reference signals from Fig. (7.6)

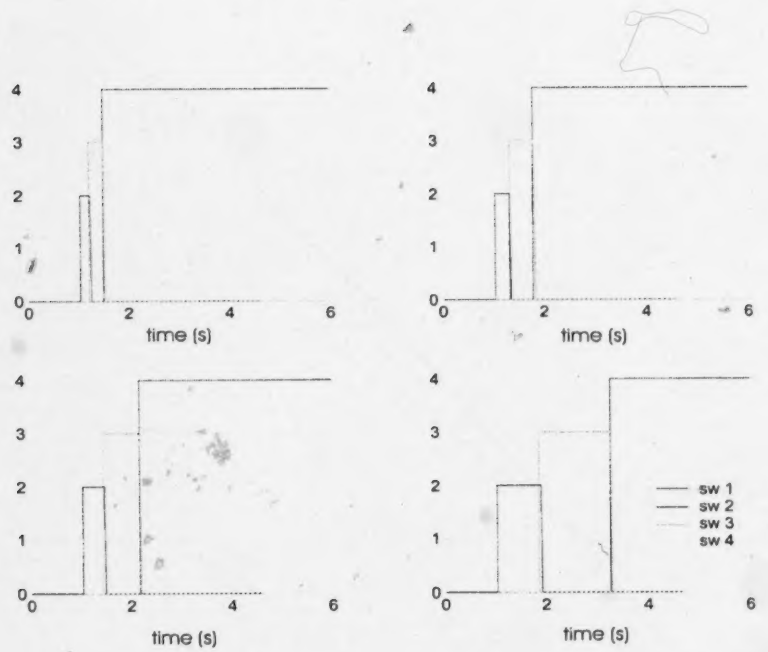

Figure 7.8: Values of the functions from (7.1) for the reference signals at Fig. (7.6)

Chapter 8

Conclusion and Future Work

In this section, we provide some concluding remarks and some ideas for further research.

8.1 Summary

The research reported in this thesis was motivated by problems related to the control of complex nonlinear systems over a wide range of operations. The idea of using multiple models and further deriving the control laws for each of them in order to achieve a wide range of operation was pursued. The control method discussed has similarities with classical gain scheduling, but the main difference is that it is based on the concept of stable motion. In order to apply the concept of stable motion, it was necessary to investigate the stability region of each multiple model.

The main contribution of the thesis is the analysis of the properties of a linear model within its stability region. Doing so requires the modeling of the additive uncertainty associated with the linear model. Research in analyzing the regions of stability and accuracy of the linear multiple models led us to the definition of measures of nonlinearity. We have defined measures of nonlinearity in the $\mathcal{L}_\infty$ sense and the $\mathcal{L}_2$ sense. For the first, we cast the problem in the context of input-to-state stability theory. For the second measure, the input-output stability analysis was applied. Both measures are very useful in control applications using robust linear control techniques. Hence, they represent a useful tool in gain scheduling problems. A case study, using the $\mathcal{L}_\infty$ measure of nonlinearity, was given in section 7.3 for a DC motor.

8.2 Future Work

Here, we discuss some of the problems encountered in the control technique presented in the thesis. We also suggest a possible directions towards the solution of these problems.

- The class of input-linear systems considered in the derivation of our measure of nonlinearity in the $\mathcal{L}_\infty$ sense is limiting. A calculation procedure for a larger class of systems (*e.g.* input-affine) should be developed.

- The above problem also imposes limitations for finding an upper-bound norm of

the input signal for the measure of nonlinearity in the $\mathcal{L}_2$ sense despite the fact that this measure is derived for the class of $input - affine$ systems.

- Both calculation procedures for measures of nonlinearity heavily depend on the choice of the Lyapunov function. A method for deriving Lyapunov functions that will provide more desirable results should be investigated. A starting point for this direction of research can be, for instance, the construction of a Lyapunov function that will provide the smallest possible nonlinearity measure in either sense over the largest possible region. Here, some of the reported algorithms for generating Lyapunov functions can be of use [81], [19], [34], [8], [35], [30], [31], [13], [62], [70], [71].

- In addition to the previous problem, calculation procedures for measures of nonlinearity should be further improved, especially for the case of the measure in the $\mathcal{L}_2$ sense. These procedures involve calculations at each point in the grid of the region of investigation and for this reason fall in to the class of NP hard problems [14], [44].

- For the control method described here, it was assumed that the states of the system are available. Further efforts are necessary in order to extend the control method for the application where this is not the case. Here, we face the problem of

observability and the existence of the separation-principle that should be further investigated (see *e.g.* [72], [10], [73], [26], [61]).

- Finally, we may say that these methods should be tested on more complex industrial systems such as The Syncrude Co-Generation Plant (see *e.g.* [27], [28], [26], [77], [78], [79], [61]).

Chapter 9

Appendix A

The following table represents values of γ_F, γ_G and $\|\xi\|_{\mathcal{L}_\infty}$ obtained for different upper bounds of the system input magnitude for three operating points. The procedure is the same as in the section (6.2).

o.p.	9	$\frac{rad}{s}$				
r_u	0.1	0.15	0.2	0.25	0.30	0.35
$\|\xi\|_\infty$	0.06	0.1	0.13	0.16	0.19	0.23
γ_F	0.25	0.37	0.50	0.62	0.74	0.87
γ_G	0.15	0.27	0.30	0.40	0.40	0.43
o.p.	14	$\frac{rad}{s}$				
$\|\xi\|_\infty$	0.07	0.12	0.16	0.19	0.23	0.27
γ_F	0.36	0.55	0.73	0.91	1.09	1.27
γ_G	0.22	0.33	0.45	0.48	0.50	0.54
o.p.	19	$\frac{rad}{s}$				
$\|\xi\|_\infty$	0.09	0.14	0.18	0.23	0.27	0.32
γ_F	0.51	0.77	1.03	1.29	1.54	1.80
γ_G	0.3	0.53	0.57	0.6	0.67	0.71

(9.1)

Chapter 10

Appendix B

The inverse Laplace transformation of a transfer function of the following form:

$$P(s) = \frac{K_1(s+a)(s+b)}{(s+a)(s+b)+K_1K} \tag{10.1}$$

is given by the following expression:

$$\begin{aligned}\mathcal{L}^{-1}(P(s)) = K_1 * \Bigg(\delta(t) + K_1 * K * \frac{e^{-1/2*(\sqrt{b^2-2*a*b+a^2-4*K_1*K}+b+a)*t}}{\sqrt{b^2-2*a*b+a^2-4*K_1*K}} \\ -K_1 * K * \frac{e^{-1/2*(-\sqrt{b^2-2*a*b+a^2-4*K_1*K}+b+a)*t}}{\sqrt{b^2-2*a*b+a^2-4*K_1*K}}\Bigg) = K_1(\delta(t)+p(t))\end{aligned} \tag{10.2}$$

The transfer function is not a strictly proper function thus a Dirac impulse has occured in inverse Laplace transformation. It's $\mathcal{L}_\infty$ norm is calculated by the following expression:

$$\mathcal{L}_\infty(P(s)) = |K_1| + \int_0^\infty |K_1 p(t)| dt \,. \tag{10.3}$$

The integral is also possible to obtain analytically, but it is an odd long expression and we will not put it here. The overall value of $\mathcal{L}_\infty$ norm can be calculated by Maple or by implementing in Matlab integration of impulse response of transfer function $P(s)$ by e.g. Simpson's Rule ([46] e.t.c).

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